•) we can tune the range of interaction

$$\omega_N(x) = N^{3\alpha}\, \omega(N^\alpha x) \qquad \alpha \in [0,1]$$

notice $\int \omega_N(x) = \int \omega(x)$

↳ $\alpha = 0$ range of interaction $O_N(1)$
"long range" as the size of the system is $O_N(1)$, all particles interact with each other

mean-field scaling

↳ $\alpha > 0$ $\qquad \omega_N \longrightarrow \delta$ - point interaction

range of interaction $\sim N^{-\alpha}$
average distance between the particles $N^{-1/3}$ $(\rho^{-1/3})$
dilute system $N^{-\alpha} \ll N^{-1/3}$

↳ $\alpha = 1$ ultra-dilute regime relevant for cold atom experiments

$$\omega_N(x) = \underset{\substack{\uparrow \\ \text{strong interactions}}}{N^3}\, \omega(Nx) \longrightarrow \text{rare interaction due to diluteness}$$

Gross-Pitaevski scaling (comment later)

4. BEC in the mean-field Bose gas

$$H_N^{MF} = \sum_{i=1}^{N} \left(-\Delta_{x_i} + V(x_i)\right) + \frac{1}{N-1} \sum_{i<j} \omega(x_i - x_j)$$

•) as before $V(x) \to \infty$ as $|x| \to \infty$, $\omega \geq 0$, $\hat{\omega} \geq 0$

4.1. Ground state energy of the MF gas

$$E_0^{MF}(N) = \inf_{\|\psi_N\| = 1} \langle \psi_N, H_N^{MF} \psi_N \rangle$$

4.1.1 Upper bound

(12)

- idea: weakly interacting system - maybe, at least energetically, a product trial state is good?

trial state: $\Psi_N^{\text{trial}}(x_1, \dots, x_N) = u(x_1) \dots u(x_N)$ for some normalized u.

Then:

$$E_0^{MF}(N) \leq N \langle u, (-\Delta + V) u \rangle + \frac{1}{N-1} \frac{N(N-1)}{2} \iint \overline{u(x)} \overline{u(y)} w(x-y) u(x) u(y)$$

$$= N \int (|\nabla u|^2 + V |u|^2) + \frac{N}{2} \iint w(x-y) |u(x)|^2 |u(y)|^2 \, dx \, dy$$

$$= N \int |\nabla u|^2 + V |u|^2 + \frac{N}{2} \int w * |u|^2 (x) |u(x)|^2 \, dx$$

we define $\mathcal{E}_H(u) = \int (|\nabla u|^2 + V |u|^2) + \frac{1}{2} \int (w * |u|^2) |u|^2$

Then $$\frac{E_0^{MF}(N)}{N} \leq \underbrace{\inf_{\|u\|_2 = 1} \mathcal{E}_H(u)}_{= e_H}$$ — Hartree functional

<u>**Thm**</u> the assumptions on V and w imply that the $\mathcal{E}_H$ admits a pointwise positive minimizer u_H which is unique up to phase and which satisfies the Euler-Lagrange equation

$$(-\Delta + V + w * |u_H|^2) u_H = \varepsilon_0 u_H$$

<u>Hartree equation</u>

ε_0: Lagrange multiplier due to the condition $\|u\| = 1$

where

$$\varepsilon_0 = \mathcal{E}_H(u_H) + \frac{1}{2} \langle u_H, w * |u_H|^2 u_H \rangle = e_H + \frac{1}{2} \langle u_H, w * |u_H|^2 u_H \rangle$$

Remarks

-) proof follows from direct method of calculus of variations

-) the Hartree equation is derived from differentiating the map $t \mapsto \mathcal{E}_H(u_{\varphi t})$, $u_{\varphi t} = \frac{u + t\varphi}{\|u + t\varphi\|_2}$, $\varphi \in C_c^\infty(\mathbb{R}^3)$ at its minimum $t=0$.
-) positivity of the solution follows basically from the inequality
$$\int |\nabla |\varphi(x)||^2 dx \leq \int |\nabla \varphi(x)|^2 dx.$$

-) let us introduce the Hartree Hamiltonian
$$H_H = -\Delta + V + w * |u_H|^2$$

this is a Schrödinger type operator with purely discrete spectrum with the ground state energy ε_0 (because, since u_H is positive, it is a ground state) other eigenvalues $\varepsilon_0 < \varepsilon_1 \leq \varepsilon_2 \leq \dots$

4.1.2 Lower bound

Lemma (Onsager)
$$\frac{1}{N-1} \sum_{i<j} w(x_i - x_j) \geq -\frac{(N-1)}{2} \iint w(x-y) |u_H(x)|^2 |u_H(y)|^2 dx\, dy$$
$$+ \sum_{i=1}^N w * |u_H|^2 (x_i) - \frac{N}{2(N-1)} w(0)$$

Proof

-) we use that $\hat{w} \geq 0 \implies \iint w(x-y) f(x) f(y) = \int \hat{w}(k) |\hat{f}(k)|^2$

and apply this for $f = |u_H|^2 - \frac{1}{N-1} \sum_i \delta_{x_i}$

Indeed, then we get
$$0 \leq \iint w(x-y) \Big(|u_H(x)|^2 - \frac{1}{N-1} \sum_i \delta(x - x_i) \Big) \Big(|u_H(y)|^2 - \frac{1}{N-1} \sum_j \delta(y - x_j) \Big)$$

$$= \iint w(x-y)\,|u_H(x)|^2\,|u_H(y)|^2 - \frac{1}{N-1}\iint w(x-y)\,|u_H(x)|^2 \sum_j \delta(y-x_j)$$ 14

$$- \frac{1}{N-1}\iint w(x-y)\,|u_H(y)|^2 \sum_i \delta(x-x_i) + \frac{1}{(N-1)^2}\iint w(x-y) \sum_{i,j} \delta(x-x_i)\,\delta(y-x_j)$$

$$= \iint w(x-y)\,|u_H(x)|^2\,|u_H(y)|^2 - \frac{2}{(N-1)} \sum_{i=1}^{N} w * |u_H|^2(x_i)$$

$$+ \underbrace{\frac{1}{(N-1)^2} \sum_{i \neq j} w(x_i - x_j)}_{= \frac{1}{(N-1)^2} \sum_{i<j} w(x_i - x_j)} + \frac{N}{(N-1)^2}\, w(0)$$

$$\Rightarrow \frac{1}{(N-1)} \sum_{i<j} w(x_i - x_j) \geq -\frac{(N-1)}{2} \iint w(x-y)\,|u_H(x)|^2\,|u_H(y)|^2$$

$$+ \sum_{i=1}^{N} w * |u_H|^2(x_i) - \frac{N}{2(N-1)}\, w(0).$$

$\square$

Using this lemma we lower bound H_N

$$H_N = \sum_{i=1}^{N} (-\Delta_{x_i} + V(x_i)) + \frac{1}{N-1} \sum_{i<j} w(x_i - x_j) \geq$$

$$\sum_{i=1}^{N} \left(-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i)\right) - \frac{N-1}{2} \iint w(x-y)\,|u_H(x)|^2\,|u_H(y)|^2$$

$$- \frac{N}{2(N-1)}\, w(0) = H_H - \frac{N}{2} \iint w(x-y)\,|u_H(x)|^2\,|u_H(y)|^2$$

$$+ \frac{1}{2} \iint w(x-y)\,|u_H(x)|^2\,|u_H(y)|^2 - \frac{N}{2(N-1)}\, w(0)$$

Since $H_H \geq N \mathcal{E}_0 = N e_H + \frac{N}{2} \iint |u_H(x)|^2 w(x-y)\,|u_H(y)|^2$

we get

$$H_N \geq N e_H + \frac{1}{2} \iint w(x-y)\,|u_H(x)|^2\,|u_H(y)|^2 - \frac{N}{2(N-1)}\, w(0)$$

In particular, since this an operator inequality, we obtain for the ground state

$$\langle \psi_N, H_N \psi_N\rangle \geq N e_H - O(1)$$

We conclude from the upper and lower bounds

<u>**Thm**</u>

$$\lim_{N\to\infty} \frac{E_0^{HF}(N)}{N} = e_H$$

<u>Remark</u> •) the above proof goes back to Grech-Seiringer (CMP 2013)
•) more general interactions and kinetic operators treated by Lewin-Nam-Rougerie (Adv. Math 2014)

<u>4.2. BEC in MF limit</u>

recall from our lower bound

$$H_N \geq \sum_{i=1}^N \left(-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i)\right) - \frac{N-1}{2}\iint w(x-y)|u_H(x)|^2|u_H(y)|^2$$

$$- \frac{N}{2(N-1)} w(0) = N\varepsilon_0 + \sum_{i=1}^N \left(-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i) - \varepsilon_0\right)$$

$$- \frac{N-1}{2}\iint w(x-y)|u_H(x)|^2|u_H(y)|^2 - \frac{N}{2(N-1)} w(0)$$

$$= N e_H + \sum_{i=1}^N (\underbrace{-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i)}_{h_H(x_i)} - \varepsilon_0) + O(1)$$

Recall $h_H = \sum_j \varepsilon_j |u_j\rangle\langle u_j|$ with $u_0 \equiv u_H$

$$\Rightarrow h_H - \varepsilon_0 = \sum_{j=1}^\infty (\varepsilon_j - \varepsilon_0)|u_j\rangle\langle u_j| \geq (\varepsilon_1 - \varepsilon_0)(1 - |u_H\rangle\langle u_H|) \geq 0$$

Thus: $\langle \psi_N, H_N^{MF} \psi_N\rangle \geq N e_H + (\varepsilon_1 - \varepsilon_0)(N - \langle u_H, \gamma_N^{(1)} u_H\rangle)$
$+ O(1)$

Here we used

$$\langle \psi_N, \sum_{i=1}^{N} \left(1 - |u_H\rangle\langle u_H|\right)_{x_i} \psi_N \rangle = N - \operatorname{Tr}\left(|u_H\rangle\langle u_H| \gamma^{(1)}_{\psi_N}\right)$$
$$= N - \langle u_H, \gamma^{(1)}_{\psi_N} u_H \rangle$$

for any normalized ψ_N. If ψ_N satisfies

$$\langle \psi_N, H_N^{MF} \psi_N \rangle \leq N e_H + \xi \quad \text{(approximate minimizer)}$$

we get

$$N e_H + \xi \geq \langle \psi_N, H_N^{MF} \psi_N \rangle \geq N e_H + (\varepsilon_1 - \varepsilon_0)\left(N - \langle u_H, \gamma^{(1)}_{\psi_N} u_H \rangle\right) + o(1)$$

$$\Downarrow$$

$$\xi + (\varepsilon_1 - \varepsilon_0) \langle u_H, \gamma^{(1)}_{\psi_N} u_H \rangle \geq (\varepsilon_1 - \varepsilon_0) N$$

$$\Longrightarrow \quad \frac{\langle u_H, \gamma^{(1)}_{\psi_N} u_H \rangle}{N} \geq 1 - \frac{\xi}{(\varepsilon_1 - \varepsilon_0) N}$$

we conclude:

Thm

The MF Bose gas displays BEC with the BEC wave-function given by the Hartree minimizer.

Alternative formulation:

$$\frac{1}{N} \gamma^{(1)}_{\psi_N} \longrightarrow |u_H\rangle\langle u_H| \quad \text{in trace class}$$