In particular, since this an operator inequality, we obtain for the ground state

$$\langle \psi_N, H_N \psi_N \rangle \geq N e_H - O(1)$$

We conclude from the upper and lower bounds

Thm

$$\lim_{N\to\infty} \frac{E_0^{HF}(N)}{N} = e_H$$

Remark •) the above proof goes back to Grech-Seiringer (CMP 2013)
•) more general interactions and kinetic operators treated by Lewin-Nam-Rougerie (Adv. Math 2014)

4.2. BEC in MF limit

recall from our lower bound

$$H_N \geq \sum_{i=1}^N (-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i)) - \frac{N-1}{2} \iint w(x-y) |u_H(x)|^2 |u_H(y)|^2$$

$$- \frac{N}{2(N-1)} w(0) = N\varepsilon_0 + \sum_{i=1}^N (-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i) - \varepsilon_0)$$

$$- \frac{N-1}{2} \iint w(x-y) |u_H(x)|^2 |u_H(y)|^2 - \frac{N}{2(N-1)} w(0)$$

$$= N e_H + \sum_{i=1}^N \big(\underbrace{-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i)}_{h_H(x_i)} - \varepsilon_0\big) + O(1)$$

Recall $h_H = \sum_j \varepsilon_j |u_j\rangle\langle u_j|$ with $u_0 \equiv u_H$

$$\Rightarrow h_H - \varepsilon_0 = \sum_{j=1}^\infty (\varepsilon_j - \varepsilon_0) |u_j\rangle\langle u_j| \geq (\varepsilon_1 - \varepsilon_0)(1 - |u_H\rangle\langle u_H|) \geq 0$$

Thus: $\langle \psi_N, H_N^{MF} \psi_N \rangle \geq N e_H + (\varepsilon_1 - \varepsilon_0)(N - \langle u_H, \gamma_{\psi_N}^{(1)} u_H \rangle) + O(1)$

Here we used

$$\langle \psi_N, \sum_{i=1}^{N} (1-|u_H\rangle\langle u_H|)_{x_i} \psi_N\rangle = N - \mathrm{Tr}\,(|u_H\rangle\langle u_H| \gamma_{\psi_N}^{(1)})$$
$$= N - \langle u_H, \gamma_{\psi_N}^{(1)} u_H\rangle$$

for any normalized ψ_N. If ψ_N satisfies

$$\langle \psi_N, H_N^{MF} \psi_N\rangle \leq N e_H + \xi \quad \text{(approximate minimizer)}$$

we get

$$N e_H + \xi \geq \langle \psi_N, H_N^{MF} \psi_N\rangle \geq N e_H + (\varepsilon_1-\varepsilon_0)(N - \langle u_H, \gamma_{\psi_N}^{(1)} u_H\rangle) + o(1)$$

$$\Downarrow$$

$$\xi + (\varepsilon_1-\varepsilon_0)\langle u_H, \gamma_{\psi_N}^{(1)} u_H\rangle \geq (\varepsilon_1-\varepsilon_0) N$$

$$\Rightarrow \quad \frac{\langle u_H, \gamma_{\psi_N}^{(1)} u_H\rangle}{N} \geq 1 - \frac{\xi}{(\varepsilon_1-\varepsilon_0)N}$$

we conclude:

Thm

The MF Bose gas displays BEC with the BEC wave-function given by the Hartree minimizer.

Alternative formulation:

$$\frac{1}{N}\gamma_{\psi_N}^{(1)} \longrightarrow |u_H\rangle\langle u_H| \quad \text{in trace class}$$

Remarks:

- the fact that the gap ~~closes~~ doesn't close, means $\mathcal{E}_1 - \mathcal{E}_0 > 0$ as $N \to \infty$. If $\mathcal{E}_1 \to \mathcal{E}_0$ then we would be in trouble.

- as in the case of the ground state energy thm, one can extend these results for more singular interactions

- In the GP scaling the same result holds but one has to replace the Hartree minimizer by the GP minimizer, i.e. the minimizer of GP functional where $w * |u|^2(x)|u(y)|^2$ term is replaced by $4\pi a|u|^4$ term.

Digression: dynamics of bosons

- imagine, we prepare a condensate in a trap, i.e. in the ground state of the ~~trapp~~ Hamiltonian with a trap. Then we turn off the trap and let the condensate evolve: what will happen to the condensate? Will the system still be condensed? In which state?

Thm
let $\psi_N(0)$ be such that $\frac{1}{N}\gamma^{(1)}_{\psi_N(0)} \longrightarrow |u_0\rangle\langle u_0|$

then $\frac{1}{N}\gamma^{(1)}_{\psi_N(t)} \longrightarrow |u_t\rangle\langle u_t|$, where

$\psi_N(t) = e^{-itH_N}\psi_N(0)$ and u_t solves the

time-dependent Hartree equation

$$i\partial_t u_t = (-\Delta + W * |u_t|^2)u_t \quad , \quad u_{t=0} = u_0$$

In the GP scaling Hartree $\to$ GP eq.

$$i\partial_t u_t = (-\Delta + 4\pi a|u_t|^2)u_t$$

- Erdős-Schlein-Yau 05-08, Pickl '08 ...

5. Bogoliubov theory

5.1. Heuristics

- 1941 Landau introduces his criterion for superfluidity: it occurs if the dispersion relation of an "elementary excitation" satisfies $\min_p \frac{\varepsilon(p)}{|p|} > 0$.

- 1947 Bogliubov gives an explanation starting from the microscopic theory:

↳ $$H = \sum_p p^2 a_p^* a_p + \frac{1}{2L^3} \sum_{p,k,q} \hat{\omega}(k)\, a_{p+k}^* a_{q-k}^* a_p a_q$$

1) macroscopic occupation of $p=0$ mode (assumed) suggests terms with less than two a_0/a_0^* operators can be neglect

2) c-number substitution: $\langle a_0^* a_0 \rangle \gg 1$ but $[a_0, a_0^*] = 1$ ⇒ we think of a_0 and a^* as commuting ⇒ classical modes ⇒ $a_0, a_0^* \sim \sqrt{N_0}$

- These two steps lead to

$$H \approx \frac{\hat{\omega}(0)\rho}{2}(N-1) + \mathbb{H}_{Bog} + R \qquad \text{where}$$

$$\mathbb{H}_{Bog} = \sum_{k \neq 0} (k^2 + \rho\hat{\omega}(k))\, a_k^* a_k + \frac{1}{2} \sum \rho\, \hat{\omega}(k) (a_k^* a_{-k}^* + a_k a_{-k})$$

$$R = \frac{-\hat{\omega}(0)}{2L^3} (N - N_0)(N - N_0 - 1) + \sum_{k \neq 0} \frac{\hat{\omega}(k)}{2L^3} \left((a_0 a_0^* {}^* - N) a_k a_k + h.c. \right)$$

$\mathbb{H}_{Bog}$ – effective quadratic Hamiltonian (so-called Bogoliubov Hamiltonian)

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•) crucial point: Bogoliubov Hamiltonian can be diagonalized:

$\exists\, \mathcal{U}_{Bog}$ - unitary on $\mathcal{F}_+$ s.t.

$$\mathcal{U}_{Bog} \mathbb{H}_{Bog} \mathcal{U}_{Bog}^* = \sum_{p \neq 0} \varepsilon(p)\, a_p^* a_p + E_{Bog}$$

with $\varepsilon(p) = p\sqrt{p^2 + 2\rho\hat{\omega}(p)}$, $E_{Bog} = -\frac{1}{2} \sum_{p \neq 0} \left(\rho\hat{\omega}(p) + p^2 - \varepsilon(p)\right)$

•) by defining $\mathcal{U}_{Bog}^* a_p^* \mathcal{U}_{Bog} =: b_p^* = (c_p a_p + s_p a_{-p}^*)$

$$\mathbb{H}_{Bog} = \sum_{p \neq 0} \varepsilon(p)\, b_p^* b_p + E_{Bog}$$

and together

$$H \approx \frac{\hat{\omega}(0)}{2} \rho\,(N-1) + E_{Bog} + \underbrace{\sum_{p \neq 0} \varepsilon(p)\, b_p^* b_p}$$

effective description in terms of non-interacting "quasi-particles"

↳ $\inf_{p} \frac{\varepsilon(p)}{p} > 0 \Rightarrow$ justification of Landau criterion

↳ description of excitation spectrum

•) Bogoliubov theory become very successful (in its variants) in many branches of physics.

•) Interesting mathematical problem: justification of Bogoliubov theory

•) also because of LHY formula (hidden in E_{Bog})

•) first proof of Bogoliubov theory

↳ MF model Seiringer 2010 (p.b.c)

↳ Lewin-Nam-Serfaty-Solovej 2015, Grech-Seiringer 2013 (trapped)
Derezinski-N 2014 (large volume)

⋮

↳ Boccato-Brennecke-Cenatiempo-Schlein 2018. p.b.c GP limit

5.2. Mathematical approach to Bogoliubov theory

- difficulties:
  - ↳ how to implement the c-number substitution
  - ↳ how to estimate higher order terms

c-number substitution

- exponential property of the Fock space

$$\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2) \simeq \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2)$$

we apply it for $\mathcal{H}_1 = P\mathcal{H}$, where $P = |u_0\rangle\langle u_0|$
$\mathcal{H}_2 = Q\mathcal{H} = \mathcal{H}_+ \equiv \{u_0\}^\perp$, $Q = 1 - P$

where u_0 - condensate wave function

Consequently, for any $\Psi_N \in \mathcal{H}^{\otimes_s N}$ we can write

$$\Psi_N = \varphi_0 u_0^{\otimes N} + \varphi_1 \otimes_s u_0^{\otimes(N-1)} + \varphi_2 \otimes_s u_0^{\otimes(N-2)} + \dots + \varphi_N$$

when $\varphi_k \in \mathcal{H}_+^{\otimes_s k}$

<u>Df</u> $U_N : \mathcal{H}^{\otimes_s N} \longrightarrow \mathcal{F}_+^{\leq N} := \mathbb{C} \oplus \mathcal{H}_+ \oplus \mathcal{H}_+^{\otimes_s 2} \oplus \dots \oplus \mathcal{H}_+^{\otimes_s N}$

↳ truncated excitation Fock space

by
$$U_N \Psi_N = \bigoplus_{j=0}^{N} Q^{\otimes j} \left(\frac{a_0^{N-j}}{\sqrt{(N-j)!}} \Psi_N \right)$$

<u>Thm</u> $U_N : \mathcal{H}^{\otimes_s N} \longrightarrow \mathcal{F}_+^{\leq N}$ is a unitary operator that satisfies

$$U_N a_0^* a_0 U_N^* = N - \mathcal{N}_+ \qquad U_N a_m^* a_n U_N^* = a_m^* a_n \qquad m,n \neq 0$$

$$U_N a_0^* a_n U_N^* = \sqrt{N - \mathcal{N}_+}\, a_n$$

⇕ $u_m, u_n \in \mathcal{H}$

□

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the operator U_N allows to transform the Hamiltonian H_N to the excited Fock space where $\mathbb{H}_{Bog}$ "lives". Indeed, we will show that

$$U_N(H_N - Ne_H)U_N^* \approx \mathbb{H}_{Bog} \to \text{(for a trapped version)}$$

Recall $H_N = \sum_{m,n\geq 0} T_{mn} a_m^* a_n + \frac{1}{2(N-1)} \sum_{mnpq\geq 0} W_{mnpq} a_m^* a_n^* a_p a_q$

<u>Lemma</u>

$$U_N(H_N - Ne_H)U_N^* = \sum_{j=0}^{4} A_j \quad \text{, where}$$

$A_0 = \frac{1}{2} W_{0000} \frac{N_+(N_+-1)}{N-1}$

$A_1 = \sum_{n\geq 1} \left(T_{0n} + W_{000n} \frac{N - N_+ - 1}{N-1} \right) \sqrt{N-N_+}\, a_n + h.c.$

$A_2 = \sum_{m,n\geq 1} \langle u_m, (h+K)u_n \rangle a_m^* a_n + \sum_{m,n\geq 1} \langle u_m, (|u_0|^2 * w + K)u_n \rangle a_m^* a_n \frac{1-N_+}{N-1}$

$\quad + \frac{1}{2} \sum_{m,n\geq 1} \left(\langle u_m, KJu_n \rangle a_n^* a_m^* \frac{\sqrt{(N-N_+)(N-N_+-1)}}{N-1} + h.c. \right)$

Here $h_H = -\Delta + V + |u_0|^2 * w - \varepsilon_0$, $K: K(x,y) = u_0(x) u_0(y) w(x-y)$

$J: \mathcal{H} \to \mathcal{H}^*$

$A_3 = \frac{1}{N-1} \sum_{mnpq\geq 1} W_{mnp0}\, a_m^* a_n^* a_p \sqrt{N-N_+} + h.c$

$A_4 = \frac{1}{2(N-1)} \sum_{mnpq\geq 1} W_{mnpq}\, a_m^* a_n^* a_p a_q$

<u>Proof</u> direct computation; e.g.

$$U_N a_m^* a_n^* a_0 a_0 U_N^* = U_N (a_m^* a_0)(a_n^* a_0) U_N^* =$$

$$= a_m^* \sqrt{N-N_+}\, a_n^* \sqrt{N-N_+} = a_m^* a_n^* \sqrt{(N-N_+-1)(N-N_+)}$$

Formally " $\frac{\sqrt{(N-\mathcal{N}_+)(N-\mathcal{N}_+-1)}}{N-1}$ " $\longrightarrow 1$ (because $\mathcal{N}_+ \sim O(1)$)

$\Rightarrow \quad A_2 \longrightarrow \mathbb{H}_{Bog}$

One needs to show other terms are small

<u>Lemma</u> Let $\Phi \in \mathcal{F}_+^{\leq N}$. Then

$$\langle \Phi, (U_N(H_N - Ne_H)U_N^* - \mathbb{H}_{Bog})\Phi \rangle \leq C\left(\frac{\mathcal{N}_+^2}{\sqrt{N}} + \frac{1}{N}\right)$$

<u>Sketch</u> <u>of</u> <u>proof</u>

-) A_1: $\quad 0 \leq \langle h_H u_0, u_n \rangle = \langle u_0, (-\Delta + V + |u_0|^2 * w - \varepsilon_0) u_n \rangle = T_{0n} + W_{000n}$

$$\Rightarrow \quad A_1 = \sum_{n \geq 1} \left(T_{0n} + W_{000n} \cdot \frac{N - \mathcal{N}_+ - 1}{N-1} \right) \sqrt{N - \mathcal{N}_+}\, a_n + h.c$$

$$= - \sum_{n \geq 1} W_{000n} \frac{\mathcal{N}_+}{N-1} \sqrt{N - \mathcal{N}_+}\, a_n + h.c$$

Cauchy-Schwarz : $\left(ab \leq \frac{1}{\varepsilon} a^2 + \varepsilon b^2 \right)$

$$\pm A_1 \leq \sum_{n \geq 1} \left(\frac{1}{\varepsilon} |W_{000n}|^2 \frac{\mathcal{N}_+}{N-1} (N - \mathcal{N}_+) \frac{\mathcal{N}_+}{N-1} + \varepsilon a_n^* a_n \right)$$

$$\leq C \frac{1}{\varepsilon} \frac{\mathcal{N}_+^2 (N - \mathcal{N}_+)}{(N-1)^2} + \varepsilon \mathcal{N}_+ \leq \frac{C}{\varepsilon} \frac{\mathcal{N}_+^2}{N} + \varepsilon \mathcal{N}_+$$

where we used

$$\sum_{n \geq 1} |W_{000n}|^2 = \sum_{n \geq 1} |\langle u_0, K u_n \rangle|^2 \leq \|K\|_{HS}^2 < \infty$$

Choosing $\varepsilon = N^{-1/2}$ gives $\pm A_1 \leq C \frac{\mathcal{N}_+^2}{\sqrt{N}}$

other terms more complicated ... $\square$

Thus, in order to show the rest is small we need to know that $\langle N_+^2 \rangle$ is small.

Note, that from BEC we know that $\langle N_+ \rangle = O(1)$

<u>Lemma</u>

Let $\psi \in \mathcal{H}^{\otimes_s N}$ be an eigenfunction of H_N with an eigenvalue $\mu_m(H_N) = N e_H + O(1)$. Then

$$\langle \psi, N_+^2 \psi \rangle \le O(1)$$

<u>Sketch of proof</u>

•) we use the Schrödinger equation:

$$H_N \Psi = \mu_m \psi$$

$$0 = \langle \psi, N_+^2 (H_N - \mu_n) + (H_N - \mu_n) N_+^2] \psi \rangle$$

$$= 2 \langle \psi, N_+ \underbrace{(H_N - \mu_n)}_{\ge H_N - N e_H - C \ge c_0 N_+ - C} N_+ \psi \rangle + \langle \psi, [[H_N, N_+] N_+] \psi \rangle$$

•) a lengthy computation

$$\pm \langle \psi, [[H_N, N_+] N_+] \psi \rangle \le C \langle \psi, (N_+ + 1)^2 \psi \rangle$$

$$\Rightarrow \quad 0 \ge c_0 \langle \psi, N_+^3 \psi \rangle - C - C \langle \psi, (N_+ + 1)^2 \psi \rangle$$

$$\langle \psi, N_+^2 \psi \rangle \le \langle \psi, \sqrt{N_+}\, N_+^{3/2} \psi \rangle \le C \underbrace{\langle \psi, N_+ \psi \rangle}_{\le C} + C \langle \psi, N_+^3 \psi \rangle$$

$$\Rightarrow \quad \langle \psi, N_+^3 \psi \rangle \le C$$

$$\Rightarrow \quad \langle \psi, N_+^2 \psi \rangle \le C$$

□

Thm V trapping, $\hat{\omega} \geq 0$. Then

- ∃ convergence of eigenvalues: for any $i = 1, 2, \ldots$ the i-th eigenvalue of H_N satisfies

$$\lim_{N\to\infty} \left(\mu_i(H_N) - N e_H - \mu_i(\mathbb{H}_{Bog}) \right) = 0$$

- ∃ convergence of eigenstates: let $\psi_N^{(i)}$ be an eigenfunction of H_N with the i-th eigenvalue $\mu_i(H_N)$. Then

$$\lim_{N\to\infty} U_N \psi_N^{(i)} = \Phi^{(i)} \qquad (\text{strongly in } \mathcal{F}_+)$$

where $\Phi^{(i)}$ is an eigenfunct. of $\mathbb{H}_{Bog}$ with the i-th eigenvalue of $\mu_i(\mathbb{H}_{Bog})$.

Sketch of proof.

i) ground state: from Hartree theory

$$\langle \psi_N^{(1)}, H_N \psi_N^{(1)} \rangle = N e_H + O(1) \underset{\text{Lemma}}{\Longrightarrow} \langle \psi_N^{(1)}, \mathcal{N}_+^2 \psi_N^{(1)} \rangle = O(1)$$

from the main operator bound

$$\langle \psi_N^{(1)}, H_N \psi_N^{(1)} \rangle = N e_H + \underbrace{\langle U_N \psi_N^{(1)}, \mathbb{H}_{Bog} U_N \psi_N^{(1)} \rangle}_{\geq \mu_1(\mathbb{H}_{Bog})} + O\!\left(\tfrac{1}{\sqrt{N}}\right)$$

upper bound: trial state

$$\tilde{\Phi}_N^{(1)} = \frac{1\!\!1^{\leq N} \Phi^{(1)}}{\| 1\!\!1^{\leq N} \Phi^{(1)} \|} \in \mathcal{F}_+^{\leq N}$$

$$\mu_1(H_N) \leq N e_H + \mu_1(\mathbb{H}_{Bog}) + o(1)$$

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•) ground state energy convergence

from above

$$\mu_1(H_N) = \langle \psi_N^{(1)}, H_N \psi_N^{(1)} \rangle = N e_H + \langle U_N \psi_N^{(1)}, \mathbb{H}_{Bog} U_N \psi_N^{(1)} \rangle + o(1)$$

and $\mu_1(H_N) \leq N e_H + \mu_1(\mathbb{H}_{Bog}) + o(1)$

$$\Rightarrow \quad \lim_{N \to \infty} \langle U_N \psi_N^{(1)}, \mathbb{H}_{Bog} U_N \psi_N^{(1)} \rangle = \mu_1(\mathbb{H}_{Bog})$$

Using that $\mu_2(\mathbb{H}_{Bog}) > \mu_1(\mathbb{H}_{Bog})$

one can deduce $U_N \psi_N^{(1)} \to \Phi^{(1)}$ from what is called the variational convergence lemma.

•) for higher eigenvectors and eigenvalues one uses the min-max principle

$\square$

6. Conclusions and outlook

•) we analyzed spectral properties of MF Bose gas by justifying Hartree and Bogoliubov theory.

•) in particular, we proved condensation with the BEC function being the Hartree minimizer.

•) the same type of results have in the last 10 years been also obtained for the GP regime

Other developments

•) in the TL limit: Lee-Huang-Yang

$$e(\rho) = 4\pi a \rho^2 \left(1 + \frac{128}{15\sqrt{\pi}} (\rho a^3)^{1/2}\right) + \text{error}$$

Here, upper bound for hard spheres still open!
Understanding Bogoliubov theory was here really important

•) MF gas at $T > 0$

Deuchert, Nam, N. exhaustive description of the Gibbs state for $T \sim N^{2/3}$

(IAMP seminar next week by Deuchert)

•) free energy expansion in TL at $T > 0$ $\lesssim (\sqrt{\rho})^2$
(Haincl, Haberberger, Nam, Seiringer, Schlein, Nam ..)

•) important for recent developments for Fermi gases.