

In particular, since this on operator inequality, we obtain for the ground state

$$\langle \psi_N | H_N \psi_N \rangle \geq N e_H - O(1)$$

We conclude from the upper and lower bounds

Thm

$$\lim_{N \rightarrow \infty} \frac{E_0^{MF}(N)}{N} = e_H$$

Remark *) the above proof goes back to Gred-Serwinger
*) more general interactions and kinetic operators treated by Lewin-Nam-Rougerie (Adv. Math 2014)

4.2. BEC in MF limit

Recall from our lower bound

$$\begin{aligned} H_N &\geq \sum_{i=1}^N (-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i)) - \frac{N-1}{2} \iint w(x-y) |u_H(x)|^2 |u_H(y)|^2 \\ &- \frac{N}{2(N-1)} w(0) = N E_0 + \sum_{i=1}^N (-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i) - \epsilon_0) \\ &- \frac{N-1}{2} \iint w(x-y) |u_H(x)|^2 |u_H(y)|^2 - \frac{N}{2(N-1)} w(0) \\ &= N e_H + \sum_{i=1}^N \underbrace{(-\Delta_{x_i} + V(x_i) + w * |u_H|^2(x_i) - \epsilon_0)}_{h_H(x_i)} + O(1) \end{aligned}$$

Recall $h_H = \sum_j \epsilon_j |u_j\rangle \langle u_j|$ with $u_0 \equiv u_H$

$$\Rightarrow h_H - \epsilon_0 = \sum_{j=1}^{\infty} (\epsilon_j - \epsilon_0) |u_j\rangle \langle u_j| \geq (\epsilon_1 - \epsilon_0) (1 - |u_H\rangle \langle u_H|) \geq 0$$



Thus: $\langle \psi_N | H_N^{MF} \psi_N \rangle \geq N e_H + (\epsilon_1 - \epsilon_0) (N - \langle u_H | \psi_N \rangle \langle \psi_N | u_H \rangle) + O(N)$

Here we used

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$$\langle \Psi_N, \sum_{i=1}^N (1 - |u_i \times u_i|)_{x_i} \Psi_N \rangle = \text{Tr} (|u_i \times u_i| \gamma_{\Psi_N}^{(1)}) \\ = N - \langle u_i, \gamma_{\Psi_N}^{(1)} u_i \rangle$$

for any normalized Ψ_N . If Ψ_N satisfies

$$\langle \Psi_N, H_N^{\text{MF}} \Psi_N \rangle \leq N\epsilon_1 + \zeta \quad (\text{approximate minimizer})$$

we get

$$N\epsilon_1 + \zeta \geq \langle \Psi_N, H_N^{\text{MF}} \Psi_N \rangle \geq N\epsilon_0 + (\epsilon_1 - \epsilon_0) (N - \langle u_i, \gamma_{\Psi_N}^{(1)} u_i \rangle + o(1))$$

\Downarrow

$$\zeta + (\epsilon_1 - \epsilon_0) \langle u_i, \gamma_{\Psi_N}^{(1)} u_i \rangle \geq (\epsilon_1 - \epsilon_0) N$$

$$\Rightarrow \frac{\langle u_i, \gamma_{\Psi_N}^{(1)} u_i \rangle}{N} \geq 1 - \frac{\zeta}{(\epsilon_1 - \epsilon_0) N}$$

we conclude:

Thm

The MF Bose gas displays BEC with the BEC wave-function given by the Hartree minimizer.

Alternative formulation:

$$\frac{1}{N} \gamma_{\Psi_N}^{(1)} \longrightarrow |u_i \times u_i| \quad \text{in trace class}$$

Remarks:

-) the fact that the gap ~~does~~ doesn't close, means $\epsilon_1 - \epsilon_0 > 0$ as $N \rightarrow \infty$. If $\epsilon_1 \rightarrow \epsilon_0$ then we would be in trouble.
-) as in the case of the ground state energy then, one can extend these results for more singular interactions
-) In the GP scaling the same result holds but one has to replace the Hartree minimizer by the GP minimizer, i.e. the minimizer of GP functional where $w * |\psi|^2(x) |\psi(y)|^2$ term is replaced by $4\pi a |\psi|^4$ term.

Digression: dynamics of bosons

•) imagine, we prepare a condensate in a trap, i.e. in the ground state of the ~~trap~~ Hamiltonian with a trap. Then we turn off the trap and let the condensate evolve: what will happen to the condensate? Will the system still be condensed? In which state?

Then

let $\psi_N(0)$ be such that $\frac{1}{N} \int \psi_N^{(0)} \rightarrow |\psi_0\rangle \langle \psi_0|$

then $\frac{1}{N} \int \psi_N^{(t)} \rightarrow |\psi_t\rangle \langle \psi_t|$, where

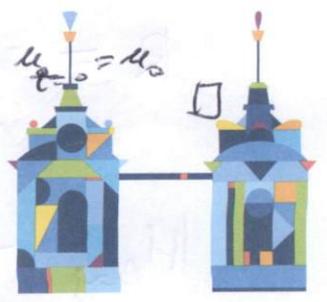
$\psi_N(t) = e^{-itH_N} \psi_N(0)$ and ψ_t solves the

time-dependent Hartree equation

$i \partial_t \psi_t = (-\Delta + W * |\psi_t|^2) \psi_t$, $\mu_{t=0} = \mu_0$

In the GP scaling Hartree \rightarrow GP eq.

$i \partial_t \psi_t = (-\Delta + 4\pi a |\psi_t|^2) \psi_t$



•) Erdős-Schlein-Yau 05-08, Pridel '03 ...

5. Bogolubov theory

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5.1. Heuristics

•) 1941 Landau introduces his criterion for superfluidity: it occurs if the dispersion relation of an "elementary excitation" satisfies $\min_p \frac{E(p)}{|p|} > 0$.

•) 1947 Bogolubov gives an explanation starting from the microscopic theory:

$$\hookrightarrow H = \sum_p p^2 a_p^\dagger a_p + \frac{1}{2L^3} \sum_{p, k, q} \tilde{\omega}(k) a_{p+k}^\dagger a_{p-k} a_p a_q$$

1) macroscopic occupation of $p=0$ mode (assumed) suggests terms with less than two a_0/a_0^\dagger operators can be neglected

2) C-number substitution: $\langle a_0^\dagger a_0 \rangle \gg 1$ but $[a_0, a_0^\dagger] = 1 \Rightarrow$ we think of a_0 and a_0^\dagger as commuting \Rightarrow classical modes $\Rightarrow a_0, a_0^\dagger \sim \sqrt{N_0}$

•) These two steps lead to

$$H \approx \frac{\tilde{\omega}(0)}{2} N(N-1) + H_{\text{Bog}} + R \quad \text{where}$$

$$H_{\text{Bog}} = \sum_{k \neq 0} (k^2 + p \tilde{\omega}(k)) a_k^\dagger a_k + \frac{1}{2} \sum_{k \neq 0} p \tilde{\omega}(k) (a_k^\dagger a_{-k}^\dagger + a_k a_k)$$

$$R = \frac{-\tilde{\omega}(0)}{2L^3} (N-N_0)(N-N_0-1) + \sum_{k \neq 0} \frac{\tilde{\omega}(k)}{2L^3} (a_0^\dagger a_0^\dagger - N_0 a_k a_k + \text{h.c.})$$

H_{Bog} - effective quadratic Hamiltonian (so-called Bogolubov Hamiltonian)

o) crucial point: Bogoliubov Hamiltonian can be diagonalized:

$\exists U_{Bog}$ - unitary on \bar{V}_+ s.t.

$$U_{Bog} H_{Bog} U_{Bog}^* = \sum_{p \neq 0} \epsilon(p) a_p^* a_p + E_{Bog}$$

with $\epsilon(p) = p \sqrt{p^2 + 2p\omega(p)}$, $E_{Bog} = -\frac{1}{2} \sum_{p \neq 0} (p\omega(p) + p^2 - \epsilon(p))$

o) by defining $U_{Bog}^* a_p^* U_{Bog} =: b_p^* = (c_p a_p + s_p a_{-p}^*)$

$$H_{Bog} = \sum_{p \neq 0} \epsilon(p) b_p^* b_p + E_{Bog}$$

and together:

$$H \approx \frac{\omega(0)}{2} (N-1) + E_{Bog} + \sum_{p \neq 0} \epsilon(p) b_p^* b_p$$

terms of non-interacting "quasi-particles" effective description in

$\hookrightarrow \inf_{p \neq 0} \frac{\epsilon(p)}{p} > 0 \Rightarrow$ justification of London criterion

\hookrightarrow description of excitation spectrum

o) Bogoliubov theory become very successful (in its variants) in many branches of physics.

o) Interesting mathematical problem: justification of Bogoliubov theory.

o) also because of LHY formulae (hidden in E_{Bog})

o) first proof of Bogoliubov theory

\hookrightarrow HF model Seiringer 2010 (p.b.c)

\hookrightarrow Lewin-Nam-Seiringer-Sobolev 2015, Guedes-Seiringer 2013

\hookrightarrow Jarekowiak-N 2014 (large volume)

\hookrightarrow Boccato-Brennecke-Cenatiempo-Schlein 2018. p.b.c GP limit



5.2. Mathematical approach to Bogoliubov theory (20)

o) difficulties: 1

↳ how to implement the c-number substitution

↳ how to estimate higher order terms

c-number substitution

o) exponential property of the Fock space

$$\tilde{F}(\mathcal{H}_1 \oplus \mathcal{H}_2) \simeq \tilde{F}(\mathcal{H}_1) \otimes \tilde{F}(\mathcal{H}_2)$$

We apply it for $\mathcal{H}_1 = P\mathcal{H}$, where $P = |u_0\rangle\langle u_0|$

$$\mathcal{H}_2 = Q\mathcal{H} = \mathcal{H}_+ \equiv \mathcal{H}_0^\perp, \quad Q = 1 - P$$

where u_0 - condensate wave function

Consequently, for any $\Psi_N \in \mathcal{H}^{\otimes N}$ we can write

$$\Psi_N = \varphi_0 u_0^{\otimes N} + \varphi_1 \otimes_5 u_0^{\otimes(N-1)} + \varphi_2 \otimes_5 u_0^{\otimes(N-2)} + \dots + \varphi_N$$

where $\varphi_k \in \mathcal{H}_+^{\otimes k}$

DF

$$U_N : \mathcal{H}^{\otimes N} \rightarrow \mathcal{F}_+^{\leq N} := \mathbb{C} \oplus \mathcal{H}_+ \oplus \mathcal{H}_+^{\otimes 2} \oplus \dots \oplus \mathcal{H}_+^{\otimes N}$$

↳ truncated excitation Fock space

by
$$U_N \bar{\Psi}_N = \sum_{j=0}^N Q^{\otimes j} \left(\frac{a_0^{N-j}}{\sqrt{(N-j)!}} \Psi_N \right)$$

Thm
that $U_N : \mathcal{H}^{\otimes N} \rightarrow \mathcal{F}_+^{\leq N}$ is a unitary operator that satisfies:

$$U_N a_0^* a_0 U_N^* = N - N_+ \quad U_N a_m^* a_n U_N^* = a_m^* a_n \quad m, n \neq 0$$

$$U_N a_0^* a_n U_N^* = \sqrt{N - N_+} a_n \quad \square \quad \downarrow \quad U_n, a_n \in \mathcal{H}$$

the operator U_N allows to transform the Hamiltonian (21)

H_N to the excited Fock space where H_{Bog} "lives". Indeed, we will show that



$$U_N (H_N - N\epsilon_H) U_N^\dagger \approx H_{\text{Bog}} \quad (\text{for a trapped version})$$

Recall $H_N = \sum_{m,n \geq 0} T_{mn} a_m^\dagger a_n + \frac{1}{2(N-1)} \sum_{m,n,p,q \geq 0} W_{mnpq} a_m^\dagger a_n^\dagger a_p a_q$

Lemma $U_N (H_N - N\epsilon_H) U_N^\dagger = \sum_{j=0}^4 A_j$, where

$$A_0 = \frac{1}{2} W_{0000} \frac{N + (N_+ - 1)}{N-1}$$

$$A_1 = \sum_{n \geq 1} \left(T_{0n} + W_{000n} \frac{N - N_+ - 1}{N-1} \right) \sqrt{N - N_+} a_n + \text{h.c.}$$

$$A_2 = \sum_{m,n \geq 1} \langle u_m, (h+k)u_n \rangle a_m^\dagger a_n + \sum_{m,n \geq 1} \langle u_m, (|u_0|^2 w + k)u_n \rangle a_m^\dagger a_n \frac{1 - N_+}{N-1} + \frac{1}{2} \sum_{m,n \geq 1} \left(\langle u_m, k_j u_n \rangle a_n^\dagger a_m^\dagger \frac{|(N - N_+)(N - N_+ - 1)|}{N-1} + \text{h.c.} \right)$$

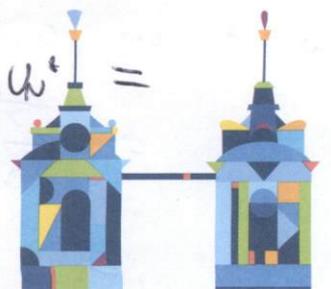
Here $h = -\Delta + V + |u_0|^2 w - \epsilon_0$, $k: \mathcal{K}(\mathbb{R}^d) \rightarrow \mathcal{K}(\mathbb{R}^d)$, $k_j: \mathcal{H} \rightarrow \mathcal{H}^\perp$

$$A_3 = \frac{1}{N-1} \sum_{m,n,p,q \geq 1} W_{mnpq} a_m^\dagger a_n^\dagger a_p a_q \sqrt{N - N_+} + \text{h.c.}$$

$$A_4 = \frac{1}{2(N-1)} \sum_{m,n,p,q \geq 1} W_{mnpq} a_m^\dagger a_n^\dagger a_p a_q$$

Proof direct computation; e.g.

$$U_N a_m^\dagger a_n^\dagger a_0 a_0 U_N^\dagger = U_N (a_m^\dagger a_0) (a_n^\dagger a_0) U_N^\dagger = a_m^\dagger \sqrt{N - N_+} a_n^\dagger \sqrt{N - N_+} = a_m^\dagger a_n^\dagger \sqrt{(N - N_+)(N - N_+)}$$



□

Formally $\left\| \frac{(N-N_+)(N-N_+-1)}{N-1} \right\| \rightarrow 1$ (because $N_+ \sim O(N)$) (22)

$\Rightarrow A_2 \rightarrow H_{Bog}$

One needs to show other terms are small

Lemma Let $\Phi \in \mathcal{F}_+^{\leq N}$ ~~Then~~ Then

$$\left\langle \Phi, (U_N (H_N - N e_u) U_N^* - H_{Bog}) \Phi \right\rangle \leq C \left(\frac{N_+^2}{N} + \frac{1}{N} \right)$$

Sketch of proof

•) A_1 : $0 \leq \langle h_u u_0, u_n \rangle = \langle u_0, (-\Delta + V + |u_0|^2 w - \varepsilon_0) u_n \rangle = T_{on} + W_{oon}$

$$\begin{aligned} \Rightarrow A_1 &= \sum_{n \geq 1} \left(T_{on} + W_{oon} \cdot \frac{N - N_+ - 1}{N-1} \right) \sqrt{N - N_+} \varrho_n + h.c \\ &= - \sum_{n \geq 1} W_{oon} \frac{N_+}{N-1} \sqrt{N - N_+} \varrho_n + h.c \end{aligned}$$

Cauchy-Schwarz: $(ab \leq \frac{1}{\varepsilon} a^2 + \varepsilon b^2)$

$$\begin{aligned} \pm A_1 &\leq \sum_{n \geq 1} \left(\frac{1}{\varepsilon} |W_{oon}|^2 \frac{N_+}{N-1} (N - N_+) \frac{N_+}{N-1} + \varepsilon \varrho_n^2 \varrho_n \right) \\ &\leq C \frac{1}{\varepsilon} \frac{N_+^2 (N - N_+)}{(N-1)^2} + \varepsilon N_+ \leq \frac{C}{\varepsilon} \frac{N_+^2}{N} + \varepsilon N_+ \end{aligned}$$

where we used

$$\sum_{n \geq 1} |W_{oon}|^2 = \sum_{n \geq 1} |\langle u_0, K u_n \rangle|^2 \leq \|K\|_{HS}^2 < \infty$$

Choosing $\varepsilon = N^{-1/2}$ gives $\pm A_1 \leq C \frac{N_+^2}{\sqrt{N}}$
 other terms were complicated. □



Thus, in order to show the rest is small we need to know that $\langle N_+^2 \rangle$ is small.

Note, that from BEC we know that $\langle N_+ \rangle = O(1)$

Lemma

Let $\psi \in \mathcal{H}^{\otimes N}$ be an eigenfunction of H_N with an eigenvalue $\mu_n(H_N) = Ne_H + O(1)$. Then $\langle \psi, N_+^2 \psi \rangle \leq O(1)$

Sketch of proof

1) we use the Schrödinger equation:

$$H_N \psi = \mu_n \psi$$

$$\begin{aligned} 0 &= \langle \psi, N_+^2 (H_N - \mu_n) + (H_N - \mu_n) N_+^2 \rangle \psi \rangle \\ &= 2 \langle \psi, N_+ (H_N - \mu_n) \psi \rangle + \langle \psi, [[H_N, N_+], N_+] \psi \rangle \\ &\geq H_N - Ne_H - C \geq c_0 N_+ - C \end{aligned}$$

2) a lengthy computation

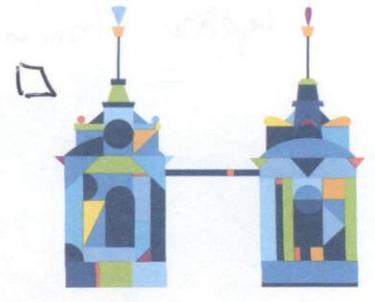
$$\pm \langle \psi, [[H_N, N_+], N_+] \psi \rangle \leq C \langle \psi, (N_+ + 1)^2 \psi \rangle$$

$$\Rightarrow 0 \geq c_0 \langle \psi, N_+^3 \psi \rangle - C - C \langle \psi, (N_+ + 1)^2 \psi \rangle$$

$$\langle \psi, N_+^2 \psi \rangle \leq \langle \psi, \sqrt{N_+} N_+^3 \psi \rangle \leq C \langle \psi, N_+ \psi \rangle + C \langle \psi, N_+^3 \psi \rangle$$

$$\Rightarrow \langle \psi, N_+^3 \psi \rangle \leq C$$

$$\Rightarrow \langle \psi, N_+^2 \psi \rangle \leq C$$



Thm V (trapping), $\hat{\omega} \geq 0$. Then

1] convergence of eigenvalues: for any $i=1,2,\dots$ the i -th eigenvalue of H_N satisfies

$$\lim_{N \rightarrow \infty} (\mu_i(H_N) - N e_H - \mu_i(H_{B_{2j}})) = 0$$

2] convergence of eigenstates: let $\psi_N^{(i)}$ be an eigenfunction of H_N with the i -th eigenvalue $\mu_i(H_N)$ then

$$\lim_{N \rightarrow \infty} U_N \psi_N^{(i)} = \underline{\Phi}^{(i)} \quad (\text{strongly in } F_+)$$

where $\underline{\Phi}^{(i)}$ is an eigenfunc. of $H_{B_{2j}}$ with the i -th eigenvalue of $\mu_i(H_{B_{2j}})$.

Sketch of proof

1) ground state: from Hartree theory

$$\langle \psi_N^{(1)}, H_N \psi_N^{(1)} \rangle = N e_H + O(1) \implies \langle \psi_N^{(1)}, M_+^2 \psi_N^{(1)} \rangle = O(1)$$

↑
lemma

from the main operator bound

$$\langle \psi_N^{(1)}, H_N \psi_N^{(1)} \rangle = N e_H + \underbrace{\langle U_N \psi_N^{(1)}, H_{B_{2j}} U_N \psi_N^{(1)} \rangle}_{\geq \mu_1(H_{B_{2j}})} + O\left(\frac{1}{N}\right)$$

upper bound:

trial state $\underline{\Phi}_N^{(1)} = \frac{\sum_{11 \in N} \underline{\Phi}^{(1)}}{\|\sum_{11 \in N} \underline{\Phi}^{(1)}\|} \in F_+ \subseteq N$

$$\mu_1(H_N) \leq N e_H + \mu_1(H_{B_{2j}}) + O(1)$$



o) ground state energy convergence

from above

$$\mu_1(H_N) = \langle \psi_N^{(1)}, H_N \psi_N^{(1)} \rangle = N\epsilon_H + \langle U_N \psi_N^{(1)}, H_{Bog} U_N \psi_N^{(1)} \rangle \quad (6.1)$$

and $\mu_1(H_N) \leq N\epsilon_H + \mu_1(H_{Bog}) + o(1)$

$$\Rightarrow \lim_{N \rightarrow \infty} \langle U_N \psi_N^{(1)}, H_{Bog} U_N \psi_N^{(1)} \rangle = \mu_1(H_{Bog})$$

Using that $\mu_2(H_{Bog}) > \mu_1(H_{Bog})$

one can deduce $U_N \psi_N^{(1)} \rightarrow \Phi^{(1)}$ from what is called the variational convergence lemma.

o) for higher eigenvectors and eigenvalues one uses the min-max principle

□

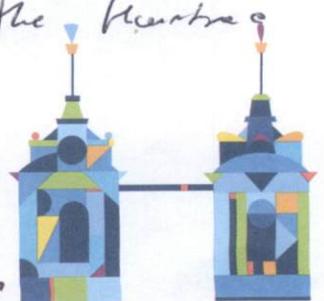
6. Conclusions and outlook

o) we analyzed spectral properties of MF Bose gas by justifying Hartree and Bogoliubov theory.

o) in particular, we proved condensation with the BEC function being the Hartree minimizer.

o) the some type of results

have in the last 10 years been also obtained for the GP regime



Other developments

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•) in the TL limit: Lee-Kuang-Yang

$$e(p) = 4\pi \rho p^2 \left(1 + \frac{128}{15\sqrt{\pi}} (\rho a^3)^{1/2} \right) + \text{error}$$

Here, upper bound for hard spheres still open!

Understanding Bogoliubov theory was here really important

•) MF gas at $T > 0$

Daudert, Nam, N. exhaustive description of the Gibbs state for $T \sim N^{2/3}$

(IAMP seminar next week by Daudert)

•) free energy expansion in TL at $T > 0 \ll (\sqrt{\rho})^2$

(Kainulainen, Kobayashi, Nam, Savary, Seiler, Nam...)

•) important for recent developments for Fermi gases.