# Momentum Distributions of Fermi Gases via Bosonization

Joint works with Niels Benedikter, Emanuela Giacomelli, Asbjørn Bækgaard Lauritsen and Diwakar Naidu

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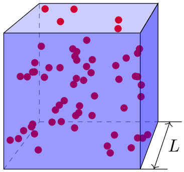
# Model and Scaling Regimes

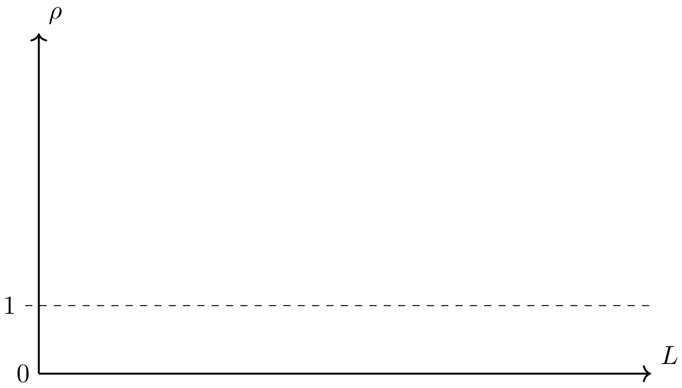
- $N$ fermions on a torus $[0,L]^3$
  $\Rightarrow$ the density is $\rho = \frac{N}{L^3}$
- Hilbert space: $\mathcal{H}^{(N)} := L^2([0,L]^3)^{\otimes_{\mathrm{a}} N}$
  That means, $\psi \in \mathcal{H}^{(N)}$ is antisymmetric
  $\psi(\ldots, x_i, \ldots, x_j, \ldots) = -\psi(\ldots, x_j, \ldots, x_i, \ldots)$.




- Hamiltonian $H_N : \mathcal{H}^{(N)} \supset \mathrm{dom}(H_N) \to \mathcal{H}^{(N)}$,

$$H_N := \sum_{j=1}^{N} -\Delta_{x_j} + \rho^{-\frac{1}{3}} \sum_{i<j}^{N} V(x_i - x_j)$$

- $(\rho, L)$ are the parameters
- Physically, we want to take $N = \rho L^3 \to \infty$.



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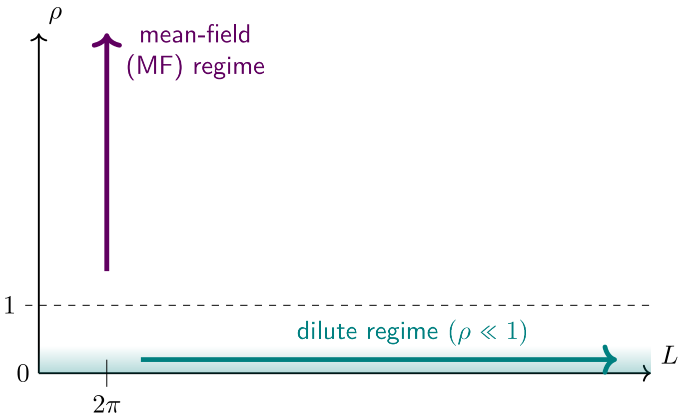

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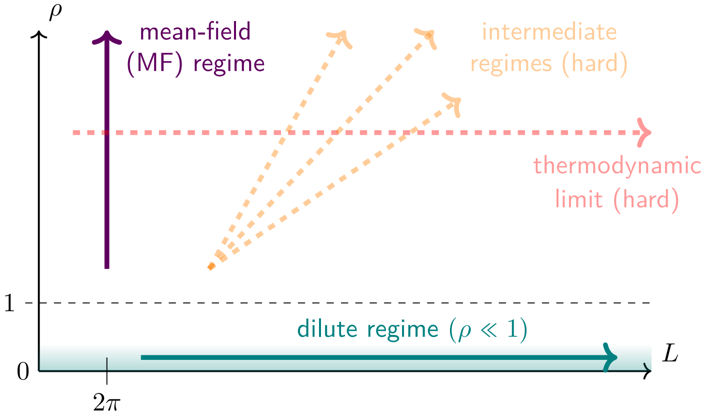
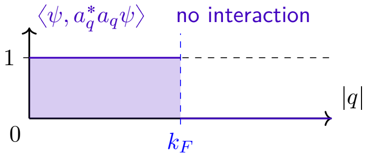
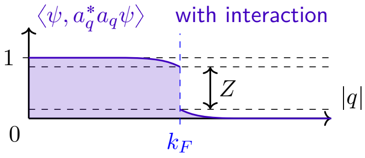


# Interesting Quantities

- **Ground state energy** $E_{\mathrm{GS}} := \inf \sigma(H_N)$, conjectured in 50's.
- **Momentum distribution** $\boxed{\langle \psi, a_q^* a_q \psi \rangle}$ for $q \in \mathbb{R}^3, \psi \approx \psi_{\mathrm{GS}}$, conjectured in 60's.




- A jump with height $Z$ is expected in $d = 3$ dimensions.
- $Z > 0$ indicates the presence of "quasiparticles", which are essential in **Fermi liquid theory** [Landau 1956-1959]

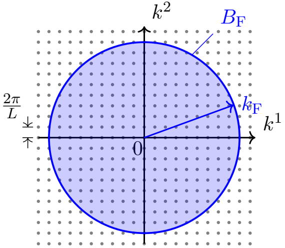
# Results

- Can Fermi gases be described as Fermi liquids?
- $d = 1$ : No, we rather have a Luttinger liquid.
- $d = 2$ : Proofs for Fermi liquid exist, using mathematical Renormalization Group (RG) techniques.
- $d = 3$ : **Open question**.

- $\exists$ very recent results on $E_{\mathrm{GS}}$ of Fermi gas (including $d = 3$) in the MF and dilute regime:
  - [Benedikter, Nam, Porta, Schlein, Seiringer 2020–24],
  - [Christiansen, Hainzl, Nam 2022–24],
  - [Fournais, Ruba, Solovej 2024],
  - [Falconi, Giacomelli, Hainzl, Porta 2020–24],
  - [Lauritsen, Seiringer 2023–24], ...

# Results - Mean Field Regime

- First approximation to GS: fill up the Fermi ball $B_{\mathrm{F}} := B_{k_{\mathrm{F}}}(0)$, s.t. $|B_{\mathrm{F}}| = N$
  $\Rightarrow$ Fermi radius $k_{\mathrm{F}} \sim \rho^{\frac{1}{3}}$
- Fermi ball state: $\psi_{\mathrm{FB}} := \bigwedge_{p_j \in B_{\mathrm{F}}} \delta_{p_j}$
- Fermi ball energy:
  $E_{\mathrm{FB}} := \langle \psi_{\mathrm{FB}}, H_N \psi_{\mathrm{FB}} \rangle \sim N \rho^{\frac{2}{3}} \sim N^{\frac{5}{3}}$




- Actually, [BNPSS] and [CHN] prove

$$E_{\mathrm{GS}} = E_{\mathrm{FB}} + E_{\mathrm{corr}} + o(N^{\frac{1}{3}}) \ ,$$

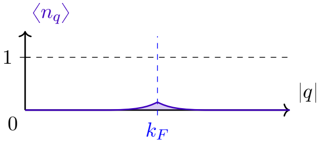
with correlation energy:

- $E_{\mathrm{corr}} \sim N^{\frac{1}{3}} \log(N)$ (for $V(x) \sim |x|^{-1}$, i.e., Coulomb) or
- $E_{\mathrm{corr}} \sim N^{\frac{1}{3}}$ (for $V$ less singular than Coulomb).

# Results - Mean Field Regime

- Define **excitation density**

$$\langle n_q \rangle := \begin{cases} 1 - \langle \psi, a_q^* a_q \psi \rangle & \text{if } q \in B_{\mathrm{F}} \\ \langle \psi, a_q^* a_q \psi \rangle & \text{if } q \notin B_{\mathrm{F}} \end{cases}$$




## Theorem ([Benedikter, L. 2023])

*For $\hat{V} \geqslant 0$ compactly supported, ∃ "RPA trial states" $(\psi_N)$ with*

$$\langle \psi_N, H_N \psi_N \rangle - E_{\mathrm{GS}} = \mathcal{O}(N^{\frac{1}{3}-\alpha}) \,, \qquad \text{while} \qquad E_{\mathrm{GS}} \sim N^{\frac{5}{3}} \,,$$

*s.t. for most $q \in \mathbb{Z}^3$,*

$$\boxed{\langle n_q \rangle = N^{-\frac{2}{3}} I(q) + \mathcal{O}(N^{-\frac{2}{3}-\frac{1}{12}})} \,,$$

*with an explicit $I(q) \sim N^0$. Further,* $\boxed{Z \geqslant 1 - \mathcal{O}(N^{-\frac{2}{3}+\frac{1}{12}})}$.

- $I(q)$ agrees with conjecture by [Daniel, Vosko 1960]

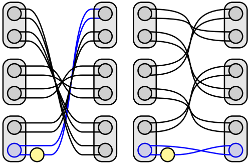
# Results - Mean Field Regime

- [L. 2024] re-establishes the formula for $n_q$ in the trial state $\psi_N$ using **Friedrichs diagrams** [Brooks, L. 2023]

## Theorem ([L. 2024])

*Under the same conditions as above,*

$$\langle n_q \rangle = \sum_{\substack{n=2 \\ n:\text{even}}}^{\infty} \sum_{\substack{G:\text{graph with} \\ n \text{ vertices}}} \mathrm{Val}(G)$$



- Expansion is exact, but *no perturbation expansion*. Leading-order terms hide in all $n$.
- $\sum_G$ and $\sum_n$ converge absolutely, but $\sum_{n,G}$ likely not.

# Results - Dilute Regime

- Switch to spin-1/2 particles, $\mathcal{H}^{(N)} = L^2([0,L]^3; \mathbb{C}^2)^{\otimes_{\mathrm{a}} N}$.
- Momentum lattice is now $\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^3$, where $L \to \infty$.
- There are 2 Fermi radii $k_{\mathrm{F}}^{\uparrow}, k_{\mathrm{F}}^{\downarrow} \sim \rho^{\frac{1}{3}}$ (which is now very small).
- The conjectured energy expansion is now

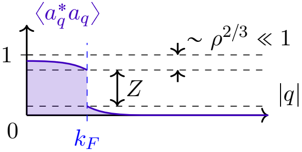
$$E_{\mathrm{GS}} = c_{\mathrm{FB}}\rho^{\frac{5}{3}} + c_{\mathrm{L}}\rho^2 + c_{\mathrm{HY}}\rho^{\frac{7}{3}} + \mathcal{O}(\rho^{\frac{8}{3}}\log(\rho)) \ .$$

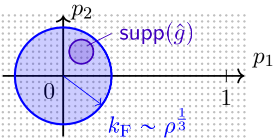
- [FGHP] and [LS] establish the $\rho^2$-term with error $\mathcal{O}(\rho^{\frac{7}{3}})$.
- [GHNS] establish upper bound including $c_{\mathrm{HY}}\rho^{\frac{7}{3}}$ with error $\mathcal{O}(\rho^{\frac{7}{3}+\frac{1}{9}})$.

# Results - Dilute Regime

- For us, smearing with $\hat{g} \in L^1(\mathbb{R}^3)$ becomes necessary, and we take the thermodynamic limit:

$$\langle n_{(g),\uparrow} \rangle := \lim_{L\to\infty} \frac{(2\pi)^3}{L^3} \sum_{p\in\Lambda^*} \hat{g}(p) \langle \psi_N, a^*_{p,\uparrow} a_{p,\uparrow} \psi_N \rangle$$




- Conjectures by [Czyż, Gottfried 1960], [Belyakov 1961], [Sartor, Mahaux 1980/82]. We consider

$$n^{(\mathrm{Bel})}_{(g),\uparrow} = \int \mathrm{d}p\ \hat{g}(p) n^{(\mathrm{Bel})}_{\uparrow}(p)\ , \qquad n^{(\mathrm{Bel})}_{\uparrow}(p) \sim \rho^{\frac{2}{3}} \ll 1$$

- $\psi_N$: trial state of [Giacomelli, Hainzl, Nam, Seiringer 2024].

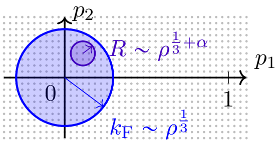
# Results - Dilute Regime

**Theorem ([Benedikter, Giacomelli, Lauritsen, L. in prep.])**

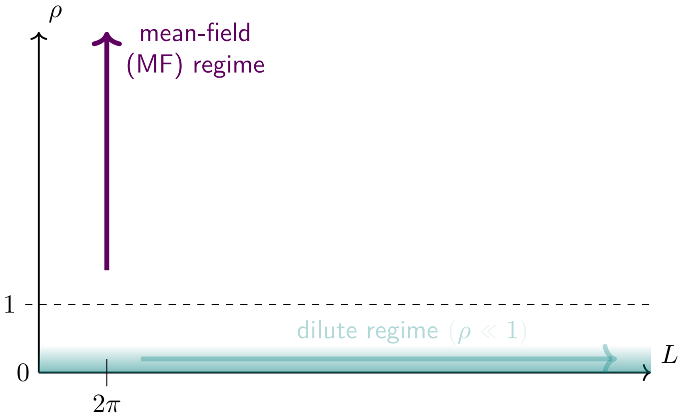
*Let $V \geqslant 0, V \in L^2$, radial, compactly supported. Smear with $\hat{g} = \chi(B_R(q))$, $R = \rho^{\frac{1}{3}+\alpha}$, $\alpha < \frac{1}{27}$. Then,*

$$\left| n^{(\mathrm{Bel})}_{(g),\uparrow} - \langle n_{(g),\uparrow} \rangle \right| \leqslant C \rho^{\frac{5}{3}+\frac{1}{9}+3\alpha} ,$$

while $\quad n^{(\mathrm{Bel})}_{(g),\uparrow} \sim \rho^{\frac{5}{3}+3\alpha}$

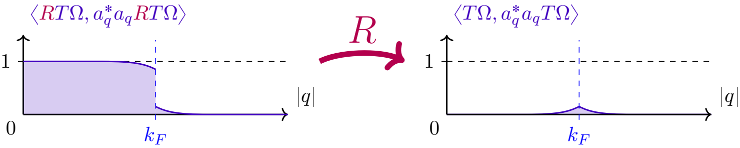


# Elements of the Proofs, Mean-Field Regime



# Elements of the Proofs, Mean-Field Regime

- We work on Fock space: $\mathcal{F} := \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}$ with vacuum $\Omega$
- Trial states are from [BNPSS]: $\boxed{\Psi_N = RT\Omega}$ with
- $R : \mathcal{F} \to \mathcal{F}$: Particle–hole transformation, $R = R^* = R^{-1}$

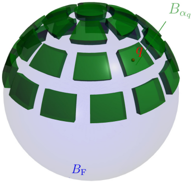


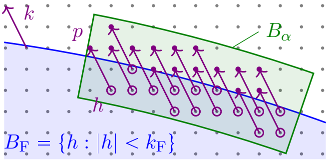

- $T : \mathcal{F} \to \mathcal{F}$: Almost-Bogoliubov transformation, $T^* = T^{-1}$

$$T := e^{-S} , \quad S := \sum_{k,\alpha,\beta} K(k)_{\alpha,\beta} \, c_\alpha^*(k) c_\beta^*(k) - \text{h.c.} ,$$

where $k, \alpha, \beta$ are indices, $K(k)_{\alpha,\beta} \in \mathbb{R}$ and
$c_\alpha^*(k), c_\alpha(k) : \mathcal{F} \to \mathcal{F}$ “bosonized operators”

- More precisely, set up $M \sim N^{\frac{1}{3}}$ patches $(B_\alpha)_{\alpha=1}^{M} \subset \mathbb{R}^3$




- Then, $c^*_\alpha(k)$, for $k \in \mathbb{Z}^3$, creates particle–hole pair:

$$c^*_\alpha(k) := \frac{1}{n_{\alpha,k}} \sum_{p,h \in \mathbb{Z}^3 \cap B_\alpha} \chi\begin{pmatrix} |p| \geqslant k_{\mathrm{F}} \\ |h| < k_{\mathrm{F}} \end{pmatrix} \delta_{h,p-k}\, a^*_p a^*_h$$

- Normalization $n_{\alpha,k} \in \mathbb{R}$ is such that "almost–CCR" hold:

$$\begin{aligned} [c_\alpha(k), c^*_{\alpha'}(k')] &= \delta_{\alpha,\alpha'}\left(\delta_{k,k'} + \mathcal{E}_\alpha(k,k')\right) , \\ \|\mathcal{E}_\alpha(k,k')\psi\| &\leqslant 2(n_{\alpha,k} n_{\alpha,k'})^{-1} \|\mathcal{N}\psi\| \qquad \forall \psi \in \mathcal{F} , \end{aligned}$$

where the number operator $\mathcal{N} = \sum_p a^*_p a_p$ is bounded as [BNPSS]: $\langle T\Omega, \mathcal{N} T\Omega \rangle = \mathcal{O}_N(1)$.

- We have to prove that:

$$\text{in general,} \qquad \langle T\Omega, a_q^* a_q T\Omega \rangle = N^{-\frac{2}{3}} I(q) + \mathcal{O}(N^{-\frac{2}{3}+\frac{1}{12}}) \,,$$

$$\text{and for most } q, \qquad \langle T\Omega, a_q^* a_q T\Omega \rangle = N^{-\frac{2}{3}} I(q) + \mathcal{O}(N^{-\frac{2}{3}-\frac{1}{12}}) \,.$$

- Idea: Apply **BCH-formula** $e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}_B^n(A)$ to $\langle T\Omega, a_q^* a_q T\Omega \rangle = \langle \Omega, e^S a_q^* a_q e^{-S} \Omega \rangle$.

- After computing and bounding many commutators, we get

$$\langle T\Omega, a_q^* a_q T\Omega \rangle = N^{-\frac{2}{3}} I(q) + \mathrm{Err} \,, \quad \text{where}$$

$$\text{in general,} \qquad \mathrm{Err} = \mathcal{O}(N^{-\frac{2}{3}+\frac{1}{12}}) + \mathcal{O}(N^{-\frac{2}{3}+\frac{1}{8}}) \Xi^{\frac{1}{2}} \,,$$

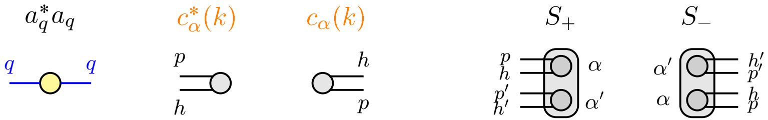
$$\text{and for most } q, \qquad \mathrm{Err} = \mathcal{O}(N^{-\frac{2}{3}-\frac{1}{12}}) + \mathcal{O}(N^{-\frac{2}{3}+\frac{1}{8}}) \Xi^{\frac{1}{2}} \,,$$

with $\Xi := \sup_{q \in \mathbb{Z}^3} \langle T\Omega, a_q^* a_q T\Omega \rangle$.

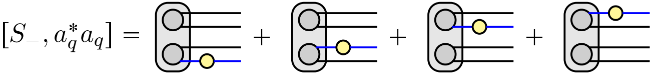
- Now, do a **bootstrap**: A priori, we know (Pauli exclusion) $\Xi \leqslant 1$. Then, iterate until $\Xi = \mathcal{O}(N^{-\frac{2}{3}+\frac{1}{12}})$. □

# Evaluation by Friedrichs Diagrams

- Evaluating $\langle n_q \rangle = \left\langle e^S a_q^* a_q e^{-S} \right\rangle_\Omega = \sum_n \frac{1}{n!} \left\langle \mathrm{ad}_S^n (a_q^* a_q) \right\rangle_\Omega$ by Friedrichs diagrams: write $S = S_+ + S_-$ and draw:

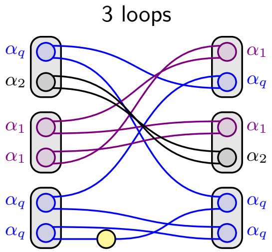


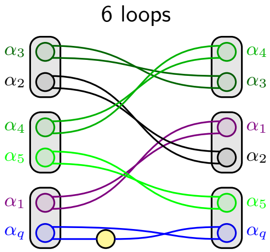

- [Brooks, L. 2023]: For 2 operators $A, B$, the commutator $[A, B]$ amounts to sum over all contractions of $\geqslant 1$ legs, e.g.,




- $\mathrm{ad}_S^n(a_q^* a_q)$ is built of $n$ vertices $S_\pm$.
- $\langle \cdot \rangle_\Omega$ eliminates all diagrams with external legs.

- The contributing diagrams are, e.g.,

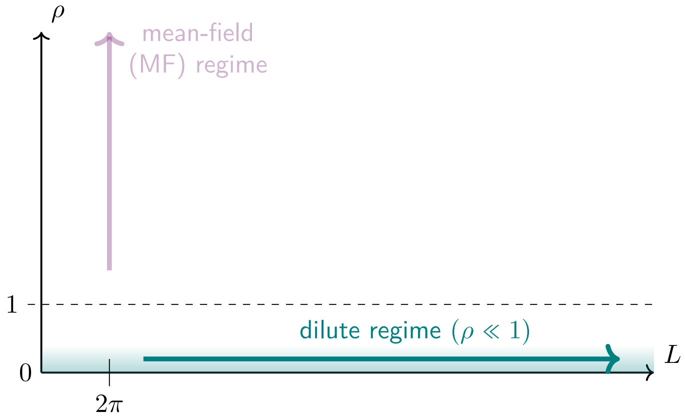



- Consist of "loops", in which the $\alpha_j$ are set equal.
- Each surviving $\sum_{\alpha_j}$ contributes a factor of $M \sim N^{\frac{1}{3}}$. Biggest diagrams are the "bosonized" ones with all loops of length 2.
- Bosonized diagrams of all even $n$ sum up to

$$\boxed{N^{-\frac{2}{3}} I(q) = \frac{1}{2} \sum_k \frac{1}{n_{\alpha_q,k}^2} \Big( \cosh(2K(k)) - 1 \Big)_{\alpha_q,\alpha_q}} \, .$$

# Elements of the Proofs, Dilute Regime

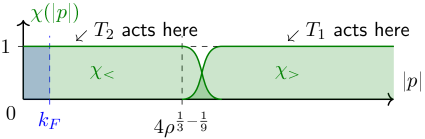


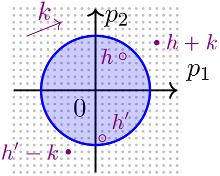
# Elements of the Proofs, Dilute Regime

- Trial states from [GHNS24]: $\boxed{\Psi_N = R T_1 T_2 \Omega}$ with

$$T_j := e^{-S_j} \;, \quad S_j := \sum_{k,h,h'} K_j(k)_{h,h'} \, b^*_{k,h,\uparrow} b^*_{-k,h',\downarrow} - \text{h.c.} \;,$$

where $k, h, h'$ are momenta and $b^*_{k,h,\sigma} := a^*_{h,\sigma} a^*_{h+k,\sigma}$



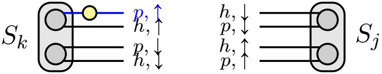

- **Coarse trafo**: $K_1(k)_{h,h'} := \hat{\varphi}(p)\chi_>(|p|)$ with $\chi_>$: IR cut-off and $\hat{\varphi}$: Fourier trafo of scattering solution
- **Fine trafo**: $K_2(k)_{h,h'} := \frac{8\pi a \chi_<(|p|)}{L^3(\Delta E(k,h,h')+\epsilon)}$ with $\chi_<$: UV cut-off, $a$: scattering length, $\epsilon > 0$, and $\Delta E(k,h,h') := \left(|h+k|^2 - |h|^2 + |h'-k|^2 - |h'|^2\right)$

- Now, diagrams get smaller as $n$ increases. It thus suffices to go to $n = 2$ by applying 2 $\times$ Duhamel on each $T_1, T_2$:

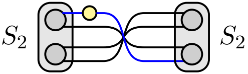
$$\left\langle n_{(g),\uparrow} \right\rangle_{\xi_1} = \int_0^1 \mathrm{d}\lambda_1 (1-\lambda_1) \left\langle [S_1, [S_1, n_{(g),\uparrow}]] \right\rangle_{\xi_\lambda} + \int_0^1 \mathrm{d}\lambda_2 \left\langle [S_2, [S_1, n_{(g),\uparrow}]] \right\rangle_{\phi_\lambda} + \int_0^1 \mathrm{d}\lambda_2 (1-\lambda_2) \left\langle [S_2, [S_2, n_{(g),\uparrow}]] \right\rangle_{\phi_\lambda} ,$$

where $\xi_\lambda := T_{2;\lambda} T_1 \Omega$ and $\phi_\lambda := T_{1;\lambda}\Omega$ with $T_{j;\lambda} := e^{\lambda S_j}$.

- For each $[S_j, [S_k, n_{(g),\uparrow}]]$ select $\geqslant 1$ out of 4 lines to contract $\Rightarrow$ 15 diagrams




fully contracted




- $n_{(g),\uparrow}^{(\mathrm{Bel})}$ "hides" in the fully contracted $S_2$-$S_2$-diagram. All other 44 diagrams are errors.

- Fully contracted $S_2$-$S_2$-diagram, without cutoffs, is

$$n_{(g),\uparrow}^{(\mathsf{Bel})} := \frac{a^2}{\pi^4} \int \mathrm{d}k \int \mathrm{d}h \int \mathrm{d}h' \frac{\hat{g}(h+k) + \hat{g}(-h)}{\Delta E(k,h,h')^2} \; .$$

- Bound 44 error diagrams as in [GHNS24] and [LS24].
- Additional complication: With $\hat{g}(p) = \delta(p-q)$ (no smearing), $\langle n_{(g),\uparrow} \rangle$ would be $\sim L^{-3}$, while errors are $\sim 1$.
  $\rightarrow$ Smearing solves this issue.
- Cutoff removal requires:

**Lemma ([BGLL, in prep.], improved Belyakov integral control)**

*In $d \geqslant 2$, for all $s \in (\frac{d}{2}, 3 + \frac{d-1}{d+1})$, $\exists C_s > 0$ s.t. $\forall x \in (0,1]$,*

$$\int_{\mathbb{R}^d} \mathrm{d}k \int_{|h|<1<|h+k|} \mathrm{d}h \int_{|h'|<x<|h'-k|} \mathrm{d}h' \frac{1}{\Delta E(k,h,h')^s} \leqslant C_s x^{-s} \; .$$

(Previously, only $s \in (\frac{d}{2}, 3)$ was known. We need $s = 3$.)

# Open Research Questions

- Finding $n_q$ for other $\psi$ and more singular potentials $V$ in MF regime: work with N. Benedikter and D. Naidu (U Milan)
- Finding $n_q$ for the true ground state
- Justifying next terms in the formulas for $n_q$ or $E_{\mathrm{GS}}$ for the MF or dilute regime
- Finding $n_q$ or $E_{\mathrm{GS}}$ in the thermodynamic limit or intermediate regimes
- ...

# Thank you for your attention!