Bose-Einstein Condensation and Spontaneous Symmetry Breaking

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20th Colloquium on Mathematics and Foundations of Quantum Theory
April 8th, 2025

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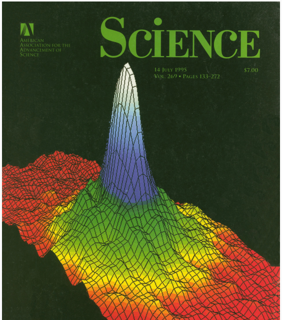
Bose-Einstein Condensation

- **1924 Bose** and **Einstein** discover a completely new type of **phase transition** in an ideal quantum gas.
- **Macroscopic occupation** of a common quantum state at low temperatures.
- Related large scale quantum effects: **superfluidity, quantized vortices**.
- Experimental realization by **Cornell, Wieman** and **Ketterle 1995**.
- 2001 Nobel Prize in Physics.



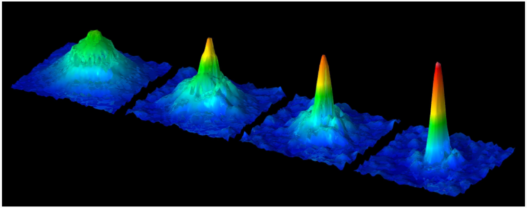
Crash course on BEC

- Consider an **ideal** gas of N bosons at $T > 0$ in a box with p.b.c.
- **Bose-Einstein distribution**:

$$\gamma^{\mathrm{id}}(p) = \frac{1}{\exp((p^2 - \mu_0)/T) - 1}$$

describes the expected number of particles with momentum p

- on the other hand $\sum_p \gamma^{\mathrm{id}}(p) = N$ (fixed by the chemical potential)
- for every $p \neq 0$ and $\mu_0 < 0$ we have $\gamma^{\mathrm{id}}(p) \to 0$ as $T \to 0$



BEC phase transition

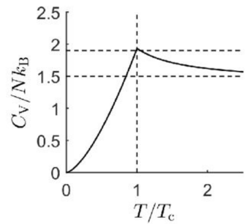
- in the **thermodynamic limit**, i.e. $N, V \to \infty$, $N/V = \rho = const$

$$\rho = \frac{1}{V}\sum_p \gamma^{\mathrm{id}}(p) \xrightarrow[V\to\infty]{} \int \gamma^{\mathrm{id}}(p)dp \leqslant \int \gamma^{\mathrm{id}}(p)\Big|_{\mu_0=0} dp =: \rho_{cr}(T)$$

- thus $\rho \leqslant \rho_{cr}(T) \xrightarrow[T\to 0]{} 0$. What is wrong?
- Below critical temperature **macroscopic occupation** of $p = 0$ mode:

$$\gamma^{\mathrm{id}}(0) = \frac{1}{\exp(-\beta\mu_0) - 1} \sim O(N)$$

- Second order **phase transition**



Some abstract nonsense...

- What about **interacting** systems?
- in **quantum statistical mechanics** the equilibrium is described by **Gibbs state**

$$G_N = \frac{e^{-\beta \mathcal{H}_N}}{\mathrm{Tr}_{\mathfrak{h}^N} e^{-\beta \mathcal{H}_N}}$$

- $\mathcal{H}_N$ - N-body Hamiltonian, $\mathfrak{h}^N$- N-body Hilbert space
- expectation values

$$\langle A \rangle = \mathrm{Tr}(A G_N)$$

- Bose-Einstein distribution is just $\langle a_p^* a_p \rangle$ for the ideal gas...
- ...which is just the diagonal of the **one-body density matrix**

$$\gamma_{G_N}^{(1)} = N \, \mathrm{Tr}_{2 \to N}[G_N]$$

Definition

A sequence of states G_N displays **Bose–Einstein condensation** iff

$$\liminf_{N \to \infty} \sup_{\|\psi\|_2 = 1} \frac{\langle \psi, \gamma_{G_N}^{(1)} \psi \rangle}{N} > 0$$

Definition of BEC

- **Many-body** Hamiltonian (again, box with p.b.c)

$$\mathcal{H}_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leqslant i < j \leqslant N} w(x_i - x_j)$$

Proving BEC in the thermodynamic limit remains open problem!

- even in the ground state!
- simpler models: **mean-field** (MF) scaling: fixed size of box

$$\mathcal{H}_N = \sum_{i=1}^{N} -\Delta_i + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} w(x_i - x_j)$$

- **BEC proven** in MF: $T = 0$ **Lieb, Seiringer 2002**, $T > 0$ **Deuchert, Seiringer 2020**

Spontaneous Symmetry Breaking (SSB)

There is SSB if a symmetry of the Hamiltonian (or Lagrangian) of a system is not present in the state under consideration (usually a ground state or a thermal equilibrium state).

- Crucial concept in Quantum Field Theory (**Nambu, Goldsotne, Higgs**...) and Statistical Physics (**Anderson, Mermin, Wagner, Hohenberg**, ...).
- **Paradigm** of statistical physics:

  Phase transitions are accompanied by SSB.

- Simple example to keep in mind: magnetization below Curie point.
- Hard to prove! Limited results: **Dyson, Lieb, Simon, Fröhlich, Spencer, Bałaban, ...** (all for lattice models)

SSB in Bose gas

- $U(1)$ symmetry of the Bose gas due to **particle number conservation**.
- second quantized Hamiltonian (momentum basis)

$$\mathcal{H} = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2|\Lambda|} \sum_{p,u,v\in\Lambda^*} \hat{w}(p) a_{u+p}^* a_{v-p}^* a_u a_v$$

- invariant under the transformation

$$a_p \mapsto e^{i\tau} a_p, \qquad \forall p$$

- Consequently, its Gibbs state has a definite number of particles.
- **Gauge invariance of a Bose gas is spontaneously broken when:**

$$\langle a_0 \rangle_{q-a} \neq 0$$

Bogoliubov quasi-averages

Bogoliubov 1961 introduces **quasi-averages** which provide a mathematical scheme how to describe SSB in statistical mechanics:

1. couple the Hamiltonian with a symmetry breaking term

$$\mathcal{H}^\lambda = \mathcal{H} + \lambda\sqrt{|\Lambda|}(a_0^* + a_0)$$

2. consider the expectation values in the perturbed Hamiltonian

$$\langle A \rangle_\lambda = \mathrm{Tr}(AG^\lambda)$$

   where

$$G^\lambda = \frac{e^{-\beta(\mathcal{H}^\lambda - \mu\mathcal{N})}}{\mathrm{Tr}\, e^{-\beta(\mathcal{H}^\lambda - \mu\mathcal{N})}}$$

3. a **quasi-average** of the observable A is then

$$\langle A \rangle_{q-a} := \lim_{\lambda\to 0} \lim_{|\Lambda|\to\infty} \langle A \rangle_\lambda$$

Grand canonical mean-field gas

- Box fixed, number of particles $N \to \infty$

$$N_0(\beta, N) = \frac{1}{\exp(-\beta\mu_0) - 1} \simeq N \left[1 - \left(\frac{\beta_\mathrm{c}}{\beta} \right)^{3/2} \right]_+ , \beta_\mathrm{c} = cN^{-2/3}$$

- Interesting parameter regime is when $T \approx N^{2/3}$.
- we will work in the grand canonical ensemble

$$\mathcal{H}_\eta = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{2\eta} \sum_{p,u,v \in \Lambda^*} \hat{w}(p) a_{u+p}^* a_{v-p}^* a_u a_v$$

- weak coupling given by $\eta \to \infty$
- scaling regime: $\beta \sim \eta^{-2/3}$, $-\eta^{2/3} \lesssim \mu \lesssim 1$
- unperturbed Gibbs state

$$G_{\beta,\mu} = \frac{\exp\left(-\beta(\mathcal{H}_\eta - \mu\mathcal{N})\right)}{\mathrm{Tr}\exp\left(-\beta(\mathcal{H}_\eta - \mu\mathcal{N})\right)},$$

- $N(\beta, \mu) = \mathrm{Tr}[\mathcal{N} G_{\beta,\mu}] =$ we prove $= c\eta + o(\eta)$

Perturbed Gibbs state

- we introduce the symmetry breaking term

$$\mathcal{H}_\eta^\lambda = \mathcal{H}_\eta + \lambda N(\beta,\mu)^{1/2}(a_0 + a_0^*)$$

- and the perturbed Gibbs state

$$G_{\beta,\mu}^\lambda = \frac{\exp\left(-\beta(\mathcal{H}_\eta^\lambda - \mu\mathcal{N})\right)}{\operatorname{Tr}\exp\left(-\beta(\mathcal{H}_\eta^\lambda - \mu\mathcal{N})\right)}.$$

- We introduce a critical temperature

$$\beta_{\mathrm{c}}(\mu,\eta) := \begin{cases} \dfrac{1}{4\pi}\left(\dfrac{\mu\,\eta}{\hat{v}(0)\zeta(3/2)}\right)^{-2/3} & \text{if} \quad \mu > 0, \\ +\infty & \text{if} \quad \mu \leqslant 0. \end{cases}$$

- Finally, we define

$$\kappa := \lim_{\eta\to\infty} \beta/\beta_{\mathrm{c}}(\mu,\eta).$$

A rigorous result...

Theorem (Deuchert, Nam, N. 2025)

Under the assumptions stated above we have

- *Bose-Einstein condensation*

$$\lim_{\eta\to\infty} \sup_{\|\psi\|=1} \frac{\langle \psi, \gamma_{\beta,\mu}\psi\rangle}{N(\beta,\mu)} = \lim_{\eta\to\infty} \frac{\mathrm{Tr}[a_0^* a_0 G_{\beta,\mu}]}{N(\beta,\mu)} = \left[1 - \frac{1}{\kappa^{3/2}}\right]_+ .$$

- *Spontaneous symmetry breaking*

$$\lim_{\lambda\to 0} \lim_{\eta\to\infty} \frac{|\,\mathrm{Tr}[a_0 G^{\lambda}_{\beta,\mu}]|}{N(\beta,\mu)^{1/2}} = \sqrt{\left[1 - \frac{1}{\kappa^{3/2}}\right]_+}.$$

- *Continuity of the condensate fraction at $\lambda = 0$*

$$\lim_{\lambda\to 0} \lim_{\eta\to\infty} \frac{|\,\mathrm{Tr}[a_0^* a_0 G^{\lambda}_{\beta,\mu}]|}{N(\beta,\mu)} = \left[1 - \frac{1}{\kappa^{3/2}}\right]_+ .$$

Relation between SSB and BEC

- as mentioned earlier **proof of BEC in the thermodynamic limit remains open**;
- quasi-averages scheme the same, but limit $V \to \infty$;
- **Lieb, Seiringer, Yngvason 2005, Süto 2005** proved that

$$\text{BEC}^{TL} \implies (\text{BEC})^{TL}_{q-a} \iff SSB^{TL}$$

- for the **mean-field model** we prove **full equivalence** which follows from the first two statements in the Theorem

$$\text{BEC}^{MF} \iff (\text{BEC})^{MF}_{q-a} \iff SSB^{MF}$$

Remarks on the proof

- the proof relies on an expansion for the grand potential

$$\Phi(\beta,\mu) = -\frac{1}{\beta}\ln\left(\operatorname{Tr}\exp(-\beta(\mathcal{H}_\eta - \mu\mathcal{N}))\right)$$

- more precisely: perturbed (by $\delta_1(a_0^* + a_0)$ and $\delta_2 a_0^* a_0$) grand potential
- we use a variational approach: $\Phi(\beta,\mu) = \min_{\Gamma\in\mathcal{S}} \mathcal{G}(\Gamma)$

$$\mathcal{G}(\Gamma) = \operatorname{Tr}[(\mathcal{H}_\eta - \mu\mathcal{N})\Gamma] - \frac{1}{\beta}S(\Gamma) \quad \text{with} \quad S(\Gamma) = -\operatorname{Tr}[\Gamma\ln(\Gamma)].$$

- upper bound: trial state

$$\Gamma^{\mathrm{trial}} = |\sqrt{N_0(\beta,\widetilde{\mu})}\rangle\langle\sqrt{N_0(\beta,\widetilde{\mu})}| \otimes G_+^{\mathrm{id}}(\beta,\widetilde{\mu})$$

- lower bound: Onsager lemma, c-number substitution and entropic inequalities
- final proof by Griffith's argument

Resume

Conclusions:

- BEC phase transition is accompanied by $U(1)$ symmetry breaking;
- we prove this fact for the mean-field Bose gas;
- we prove BEC and SSB are equivalent.

Outlook

- superfluidity, i.e. $\langle a_p a_{-p} \rangle$;
- relation between superfluidity and BEC;
- thermodynamic limit :)

Thank you for your attention!