IR-Catastrophe in the Time-Dependent van Hove Model

Fabian Nolte
Dirk Deckert

Time-Dependent
van Hove Model

fixed trajectory $r : \mathbb{R} \to \mathbb{R}^3$ of $\varrho \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$ particle + dynamical scalar quantum potential

Time-Dependent
van Hove Model

fixed trajectory $r: \mathbb{R} \to \mathbb{R}^3$ of $\varrho \in C_c^\infty(\mathbb{R}^3;\mathbb{R})$ particle + dynamical scalar quantum potential

Dynamics:

- $\mathcal{F} := \bigoplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbb{R}^3;\mathbb{C})$
- $H: \mathbb{R} \to \mathcal{L}(\mathcal{F}),\ t \mapsto H(t) := d\Gamma(\omega) + \Phi(f(t))$

$\omega_k = \sqrt{\|k\|^2 + m^2}$, $m \geq 0$, $\Phi(g) = \overline{a(g) + a^*(g)}$,

$f(t)(k) = \frac{(2\pi)^{-3/2}}{\sqrt{2\omega_k}} \left((2\pi)^{3/2} \hat{\varrho}^*(k) \right) e^{-i\langle k, r(t)\rangle}$,

(H) :$\Leftrightarrow$ $\left(\hat{\varrho}/\omega \in L^2(\mathbb{R}^3;\mathbb{C}) \right)$

Time Evolution

$$H(t) := d\Gamma(\omega) + \Phi(f(t)) \overset{(H)}{\Rightarrow} \operatorname{dom}(H(t)) = \operatorname{dom}(d\Gamma(\omega))$$

$$(U) :\Leftrightarrow \left(\hat{\varrho}\,\|\cdot\|^2/\sqrt{\omega},\ \hat{\varrho}\,\|\cdot\|/\omega \in L^2(\mathbb{R}^3; \mathbb{C}) \right)$$

$$(U) + [\text{Kato}, 53] \Rightarrow \exists!\ \text{evolution}\ \{U(t,s)\}_{t,s\in\mathbb{R}}$$

Time Evolution

$$H(t) := d\Gamma(\omega) + \Phi(f(t)) \overset{(H)}{\Rightarrow} \mathrm{dom}(H(t)) = \mathrm{dom}(d\Gamma(\omega))$$

$$(U) :\Leftrightarrow \left(\hat{\varrho}\,\|\cdot\|^2/\sqrt{\omega},\ \hat{\varrho}\,\|\cdot\|/\omega \in L^2(\mathbb{R}^3, \mathbb{C}) \right)$$

$$(U) + [\text{Kato}, 53] \Rightarrow \exists!\ \text{evolution} \quad \{U(t,s)\}_{t,s\in\mathbb{R}}$$

Explicit Form

$$\left(\begin{array}{c} \hat{\varrho}\sqrt{\omega} \in L^2(\mathbb{R}^3, \mathbb{C}) \\ \wedge\ r \in C(\mathbb{R}, \mathbb{R}^3) \end{array} \right) :\Leftrightarrow (E) \rightarrow U(t,s) = U_0(t,s)\, e^{-i\alpha(t,s)} e^{-i\Phi(F(t,s))}$$

$$F(t,s) := \int_s^t dt'\, e^{-i\omega(s-t')} f(t') \in L^2(\mathbb{R}^3, \mathbb{C}), \quad \alpha(t,s) \in \mathbb{C}$$

Field Operator

$\Phi \in LF[C_c^\infty(\mathbb{R}^3;\mathbb{C}), \mathcal{L}(\mathcal{F})]$

$$\sigma \mapsto \Phi[\sigma] := a(\hat{\sigma}^*(-\cdot)/\sqrt{2\omega}) + a^+(\hat{\sigma}(\cdot)/\sqrt{2\omega})$$

$$\text{"}=\text{"} \int d^3x \left(\int d^3k \frac{(2\pi)^{-3/2}}{\sqrt{2\omega_k}} \left(a_k e^{i\langle k,x\rangle} + a_k^+ e^{-i\langle k,x\rangle} \right) \right) \sigma(x) \text{"}$$

Field Operator

$\Phi \in LF[C_c^\infty(\mathbb{R}^3, \mathbb{C}), \mathcal{L}(\mathcal{F})]$

$$\sigma \mapsto \Phi[\sigma] := a(\hat{\sigma}^*(-\cdot)/\sqrt{2\omega}) + a^+(\hat{\sigma}(\cdot)/\sqrt{2\omega})$$

$$\text{"}=\text{"} \int d^3x \left(\int d^3k \frac{(2\pi)^{-3/2}}{\sqrt{2\omega_k}} \left(a_k e^{i\langle k,x\rangle} + a_k^+ e^{-i\langle k,x\rangle} \right) \right) \sigma(x) \text{"}$$

$\Phi_e : (\mathbb{R} \times \mathbb{R}) \longrightarrow LF[C_c^\infty(\mathbb{R}^3, \mathbb{C}), \mathcal{L}(\mathcal{F})]$

(H) + (U)

$$(t,s) \mapsto \Phi_e(t,s) := \left| \begin{array}{l} \sigma \mapsto \Phi_e(t,s)[\sigma] := U(s,t)\Phi[\sigma]U(t,s) \\ \quad = U_0(s,t)\Phi[\sigma]U_0(t,s) + \int d^3x\, \varphi_{(t,s)}(x)\, \sigma(x) \end{array} \right|$$

(E)

Field Operator

$$\Phi \in LF[C_c^\infty(\mathbb{R}^3, \mathbb{C}), \mathcal{L}(\mathcal{F})]$$

$$\sigma \mapsto \Phi[\sigma] := a\left(\hat{\sigma}^*(-\cdot)/\sqrt{2\omega}\right) + a^\dagger\left(\hat{\sigma}(\cdot)/\sqrt{2\omega}\right)$$

$$\text{"} = \int d^3x \left(\int d^3k \frac{(2\pi)^{-3/2}}{\sqrt{2\omega_k}} \left(a_k e^{i\langle k, x\rangle} + a_k^\dagger e^{-i\langle k, x\rangle} \right) \right) \sigma(x) \text{"}$$

$$\Phi_e : (\mathbb{R} \times \mathbb{R}) \longrightarrow LF[C_c^\infty(\mathbb{R}^3, \mathbb{C}), \mathcal{L}(\mathcal{F})]$$

$$(t,s) \mapsto \Phi_e(t,s) := \left| \begin{aligned} \sigma \mapsto \Phi_e(t,s)[\sigma] &\overset{(H)+(U)}{:=} U(s,t)\,\Phi[\sigma]\,U(t,s) \\ &\overset{(E)}{=} U_0(s,t)\,\Phi[\sigma]\,U_0(t,s) + \int d^3x\, \varphi_{(t,s)}(x)\, \sigma(x) \end{aligned} \right|$$

$$U_0(s,t)\,\Phi[\sigma]\,U_0(t,s)$$

$$\text{"} = \int d^3x \left(\int d^3k \frac{(2\pi)^{-3/2}}{\sqrt{2\omega_k}} \left(a_k e^{-i(\omega(t-s) - \langle k, x\rangle)} + a_k^\dagger e^{i(\omega(t-s) - \langle k, x\rangle)} \right) \right) \sigma(x) \text{"}$$

Free Field

Dressing of Φ

$$\Phi_e(t,s) = \Phi_0(t,s) + \varphi(t,s) \quad , \quad t \geq s$$

$(H) + (U) + (E)$

$\varphi(t,s)$

$$= \mathcal{F}^{-1}\left(\frac{-i}{\sqrt{2\omega_k}} \left(e^{-i\omega_\cdot(t-s)} F(t,s)(\cdot) - e^{i\omega_\cdot(t-s)} F(t,s)^*(-\cdot) \right)\right)$$

$$= \mathcal{F}^{-1}\left(\int_{-\infty}^{\infty} dt' \left((2\pi)^{-3/2} \left(+\Theta(+(t-t'))\right) \frac{\sin(\omega(t-t'))}{\omega} \right) (2\pi)^{3/2}\, \hat{\varrho}\, (2\pi)^{3/2} \left((2\pi)^{-3/2}\, \Theta(t'-s)\, e^{-i\langle k, r(t')\rangle} \right)\right)$$

Dressing of Φ

$$\Phi_e(t,s) \underset{(H)+(U)+(E)}{=} \Phi_0(t,s) + \varphi(t,s) \quad , \quad t \geq s$$

$\varphi(t,s)$

$$= \mathcal{F}^{-1}\left(\frac{-i}{\sqrt{2\omega_k}} \left(e^{-i\omega_\cdot(t-s)} F(t,s)(\cdot) - e^{i\omega_\cdot(t-s)} F(t,s)^*(-\cdot) \right) \right)$$

$$= \mathcal{F}^{-1}\left(\int_{-\infty}^{\infty} dt' \Big(\underbrace{(2\pi)^{-3/2} (+\Theta(+(t-t'))) \frac{\sin(\omega(t-t'))}{\omega}}_{= \mathcal{F}(G^{ret}(t-t',\cdot))} \Big) (2\pi)^{3/2} \hat{\varrho} \, (2\pi)^{3/2} \Big(\underbrace{(2\pi)^{-3/2} \Theta(t'-s) e^{-i\langle k, r(t')\rangle}}_{= \mathcal{F}(\Theta(t'-s)\delta(\cdot - r(t')))} \Big) \right)$$

$$"=" \left(G^{ret} *_4 \Theta(\cdot - s)\delta(\cdot - r(\cdot)) \right) *_3 \varrho \, (t)$$

Dressing of Φ

$$\Phi_e(t,s) \underset{\nearrow}{=} \Phi_0(t,s) + \varphi(t,s) \quad , \quad t \geq s$$

(H) + (U) + (E)

$\varphi(t,s)$

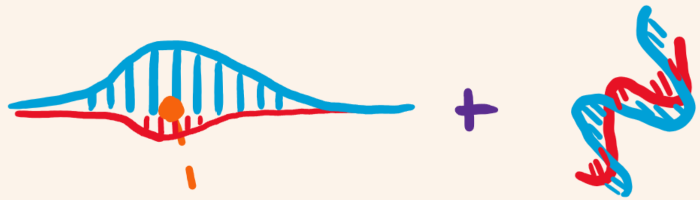
$$= \mathcal{F}^{-1}\left(\frac{-i}{\sqrt{2\omega_\cdot}} \left(e^{-i\omega_\cdot(t-s)} F(t,s)(\cdot) - e^{i\omega_\cdot(t-s)} F(t,s)^*(-\cdot) \right)\right)$$

$$= \mathcal{F}^{-1}\left(\int_{-\infty}^{\infty} dt' \left(\underbrace{(2\pi)^{-3/2} (+\Theta(+(t-t'))) \frac{\sin(\omega(t-t'))}{\omega}}_{= \mathcal{F}(G^{ret}(t-t',\cdot))} \right) (2\pi)^{3/2} \hat{\varrho}\, (2\pi)^{3/2} \left(\underbrace{(2\pi)^{-3/2} \Theta(t'-s)\, e^{-i\langle k, r(t')\rangle}}_{= \mathcal{F}(\Theta(t'-s)\delta(\cdot - r(t')))} \right)\right)$$

$$\text{"="} \underbrace{\left(G^{ret} *_4 \Theta(\cdot - s)\delta(\cdot - r(\cdot)) \right)}_{=: \varphi_\delta(t,s)(x)} *_3 \varrho\ (t)$$

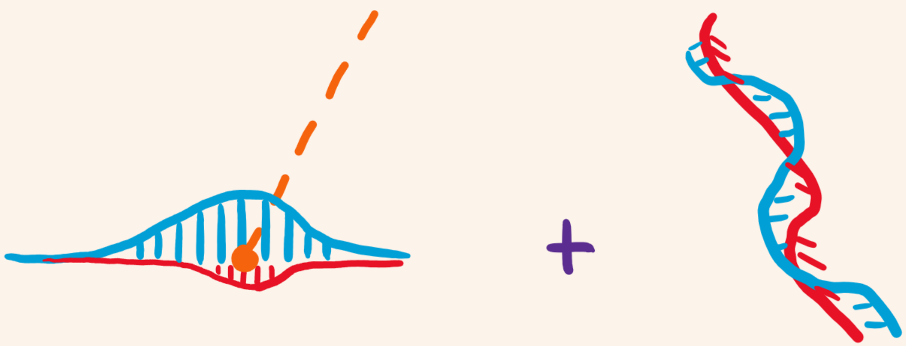
$$(m=0) \rightarrow \underset{\uparrow}{=} \frac{1}{1 + n^+(t,x)\cdot \dot{r}^+(t,x)} \frac{1}{4\pi \|x - r^+(t,x)\|} \mathbb{1}_{\Gamma^{light,+}_{graph(r|_{(-\infty,s]})}}(t,x)$$

$$(D) :\Leftrightarrow \left(\exists v_{max} < 1 \text{ s.t. } \forall t \in \mathbb{R}\ \|\dot{r}(t)\| \leq v_{max} \right)$$



Scattering situation

$$(A) :\Longleftrightarrow \left(\begin{array}{c} \exists\, T_{\pm} \in \mathbb{R} \ \text{ and } \ x_{\pm},\, v_{\pm} \in \mathbb{R}^3 \ \ (\|v_{\pm}\| < 1) \\ \text{s.t. } \forall\, t \gtrless T_{\pm} \quad r(t) = x_{\pm} + v_{\pm} t \end{array} \right)$$



Asymptotes (m=0)

$$\varphi_\delta(t,s)(x) = \left(G^{ret} *_4 \theta(\cdot - s)\delta(\cdot - r(\cdot))\right)(t,x)$$

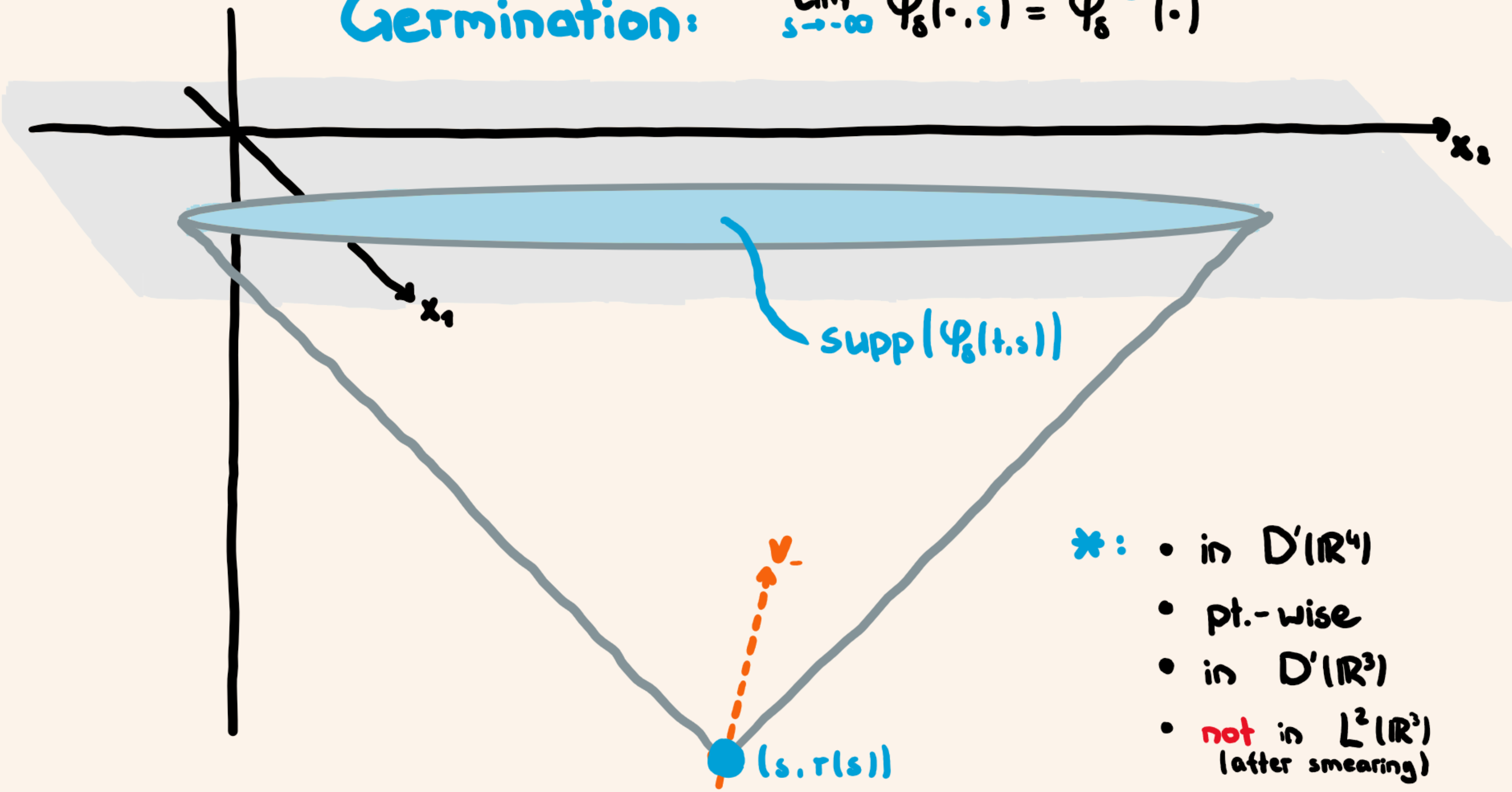
↳ $\varphi_\delta(t,-\infty) \triangleq$ Coulomb pot. (boosted by v_-)

Asymptotes $(m=0)$

$$\varphi_\delta(t,s)(x) = \left(G^{ret} *_4 \Theta(\cdot-s)\,\delta(\cdot-r(\cdot))\right)(t,x)$$

↳ $\varphi_\delta(t,-\infty) \triangleq$ Coulomb pot. (boosted by v_-)

Germination: $\lim_{s\to-\infty} \varphi_\delta(\cdot,s) \overset{*}{=} \varphi_\delta^{ret}(\cdot)$



$*$:
- in $\mathcal{D}'(\mathbb{R}^4)$
- pt.-wise
- in $\mathcal{D}'(\mathbb{R}^3)$
- not in $L^2(\mathbb{R}^3)$ (after smearing)

Wave Operator Ω

$$\Omega(t,0) \overset{(H)+(U)}{:=} U(0,t)\,U_0(t,0) \overset{(E)}{=} e^{-i\alpha(t,0)}\, e^{-i\Phi(F(t,0))}$$

Try $\lim_{t\to\pm\infty} \Omega(t,0)$ (A)

- $F(\pm\infty,0) := \mathcal{D}'(\mathbb{R}^3)\text{-}\lim_{t\to\pm\infty} F(t,0)$ $\begin{cases} \in L^2(\mathbb{R}^3,\mathbb{C}) & m>0 \\ \notin L^2(\mathbb{R}^3,\mathbb{C}) & m=0 \end{cases}$ IR-Cat.
- $m>0$: $L^2(\mathbb{R}^3)\text{-}\lim_{t\to\pm\infty} F(t,0)$ not existing
- $\mathbb{C}\text{-}\lim_{t\to\pm\infty} \alpha(t,0) = \pm\infty$

Wave Operator Ω

$$\Omega(t,0) \overset{(H)+(U)}{:=} U(0,t)\,U_0(t,0) \overset{(E)}{=} e^{-i\alpha(t,0)}\, e^{-i\Phi(F(t,0))}$$

Try $\lim_{t\to\pm\infty} \Omega(t,0)$ (A)

- $F(\pm\infty,0) := \underset{t\to\pm\infty}{D'(\mathbb{R}^3)\text{-}\lim} F(t,0) \begin{cases} \in L^2(\mathbb{R}^3,\mathbb{C}) & m>0 \\ \notin L^2(\mathbb{R}^3,\mathbb{C}) & m=0 \end{cases}$ IR-Cat.
- $m>0$: $\underset{t\to\pm\infty}{L^2(\mathbb{R}^3)\text{-}\lim} F(t,0)$ not existing
- $\underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim}\, \alpha(t,0) = \pm\infty$

Not even $\underset{\substack{t_+\to+\infty \\ t_-\to-\infty}}{\mathbb{C}\text{-}\lim} |\langle 0|\, \Omega^+(t_-,0)\,\Omega(t_+,0)\,|0\rangle|$ exists

Ω_ε Convergence Boost

$$f_\varepsilon(t) := f(t)\, e^{-\varepsilon|t|} \qquad H_\varepsilon(t) \overset{(H)}{:=} d\Gamma(\omega) + \Phi(f_\varepsilon(t))$$

$$\Omega_\varepsilon(t,0) \overset{(U)}{:=} U_\varepsilon(0,t)\, U_0(t,0) \overset{(E)}{=} e^{-i\alpha_\varepsilon(t,0)}\, e^{-i\Phi(F_\varepsilon(t,0))}$$

Ω_ε Convergence Boost

$$f_\varepsilon(t) := f(t)\, e^{-\varepsilon|t|} \qquad H_\varepsilon(t) \overset{(H)}{:=} d\Gamma(\omega) + \Phi(f_\varepsilon(t))$$

$$\Omega_\varepsilon(t,0) \overset{(U)}{:=} U_\varepsilon(0,t)\,U_0(t,0) \overset{(E)}{=} e^{-i\alpha_\varepsilon(t,0)}\, e^{-i\Phi(F_\varepsilon(t,0))}$$

Try $\lim_{t\to\pm\infty} \Omega_\varepsilon(t,0)$ (A)

- $F_\varepsilon(\pm\infty,0) := L^2(\mathbb{R}^3)\text{-}\lim_{t\to\pm\infty} F_\varepsilon(t,0)$ exists
- $\alpha_\varepsilon(\pm\infty,0) := \mathbb{C}\text{-}\lim_{t\to\pm\infty} \alpha_\varepsilon(t,0)$ exists

$\hookrightarrow$ $\Omega_\varepsilon(\pm\infty,0) := \text{strong-}\lim_{t\to\pm\infty} \Omega_\varepsilon(t,0)$ exists

Ω_ε Convergence Boost

$$f_\varepsilon(t) := f(t)\, e^{-\varepsilon|t|} \qquad H_\varepsilon(t) \overset{(H)}{:=} d\Gamma(\omega) + \Phi(f_\varepsilon(t))$$

$$\Omega_\varepsilon(t,0) \overset{(U)}{:=} U_\varepsilon(0,t)\, U_0(t,0) \overset{(E)}{=} e^{-i\alpha_\varepsilon(t,0)}\, e^{-i\Phi(F_\varepsilon(t,0))}$$

Try $\lim_{t\to\pm\infty} \Omega_\varepsilon(t,0)$ (A)

- $F_\varepsilon(\pm\infty,0) := L^2(\mathbb{R}^3)\text{-}\lim_{t\to\pm\infty} F_\varepsilon(t,0)$ exists
- $\alpha_\varepsilon(\pm\infty,0) := \mathbb{C}\text{-}\lim_{t\to\pm\infty} \alpha_\varepsilon(t,0)$ exists

$\Omega_\varepsilon(\pm\infty,0) := \text{strong-}\lim_{t\to\pm\infty} \Omega_\varepsilon(t,0)$ exists

IR
$m>0$

$$\Rightarrow \Omega(\pm\infty,0) := \text{strong-}\lim_{\varepsilon\to 0} e^{i\alpha_\varepsilon(\pm\infty,0)}\, \Omega_\varepsilon(\pm\infty,0) = e^{-i\Phi(F(\pm\infty,0))}$$

Ω_k 0-Mode Cutting

$$h_k(t) := \hat{e}/\sqrt{2W}\ \mathbb{1}_{B_k(0)} \frac{1}{w - \langle \cdot, v_{\mathrm{sgn}(t)} \rangle} \qquad H_k(t) \overset{(m>0)}{\underset{(H)}{:=}} W(h_k(t))\, H(t)\, W^{+}(h_k(t))$$

$$\Omega_k(t,0) \overset{(U)}{:=} U_k(0,t)\, U_0(t,0) \overset{(E)}{=} e^{-i\alpha_k(t,0)}\, e^{-i\Phi(F_k(t,0))}\, e^{-i\beta_k(t,0)}$$

Ω_k 0-Mode Cutting

$$h_k(t) := \hat{e}/\sqrt{2w}\ \mathbb{1}_{B_k(0)} \frac{1}{w - \langle \cdot, v_{\mathrm{sgn}(t)} \rangle} \qquad H_k(t) \overset{(m>0)}{\underset{(H)}{:=}} W(h_k(t))\, H(t)\, W^+(h_k(t))$$

$$\Omega_k(t,0) \overset{(U)}{:=} U_k(0,t)\, U_0(t,0) \overset{(E)}{=} e^{-i\alpha_k(t,0)}\, e^{-i\Phi(F_k(t,0))}\, e^{-i\beta_k(t,0)}$$

Try $\lim\limits_{t\to\pm\infty} \Omega(t,0)$ (A)

- $F_k(\pm\infty,0) := \mathcal{D}'(\mathbb{R}^3)\text{-}\lim\limits_{t\to\pm\infty} F_k(t,0) \in L^2(\mathbb{R}^3,\mathbb{C})$ $\begin{cases} m>0 \\ m=0 \end{cases}$ IR-finite
- $L^2(\mathbb{R}^3)\text{-}\lim\limits_{t\to\pm\infty} F_k(t,0)$ not existing
- $\mathbb{C}\text{-}\lim\limits_{t\to\pm\infty} \alpha_k(t,0) = \pm\infty$
- $\mathbb{C}\text{-}\lim\limits_{t\to\pm\infty} \beta_k(t,0) = \pm\infty$

Ω_κ 0-Mode Cutting

$$h_\kappa(t) := \hat{e}/\sqrt{2\omega}\ \mathbb{1}_{B_\kappa(0)} \frac{1}{\omega - \langle \cdot, v_{\mathrm{sgn}(t)}\rangle} \qquad H_\kappa(t) \overset{(m>0)}{\underset{(H)}{:=}} W(h_\kappa(t))\, H(t)\, W^+(h_\kappa(t))$$

$$\Omega_\kappa(t,0) \overset{(U)}{:=} U_\kappa(0,t)\,U_0(t,0) \overset{(E)}{=} e^{-i\alpha_\kappa(t,0)}\, e^{-i\Phi(F_\kappa(t,0))}\, e^{-i\beta_\kappa(t,0)}$$

Try $\lim\limits_{t\to\pm\infty} \Omega(t,0)$ (A)

- $F_\kappa(\pm\infty,0) := \underset{t\to\pm\infty}{D'(\mathbb{R}^3)\text{-}\lim} F_\kappa(t,0) \in L^2(\mathbb{R}^3,\mathbb{C})$ $\begin{cases} m>0 \\ m=0 \end{cases}$ IR-finite
- $\underset{t\to\pm\infty}{L^2(\mathbb{R}^3)\text{-}\lim} F_\kappa(t,0)$ not existing
- $\underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim} \alpha_\kappa(t,0) = \pm\infty$
- $\underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim} \beta_\kappa(t,0) = \pm\infty$

Not even $\underset{\substack{t_+\to+\infty \\ t_-\to-\infty}}{\mathbb{C}\text{-}\lim} |\langle 0|\, \Omega_\kappa^+(t_-,0)\, \Omega_\kappa(t_+,0)\, |0\rangle|$ **exists**

$\Omega_{\kappa,\varepsilon}$ Convergence Boost + 0-Mode Cutting

$$h_{\kappa,\varepsilon}(t) := \hat{e}/\sqrt{2\omega}\ \mathbb{1}_{B_\kappa(0)} \frac{1}{\omega - \langle \cdot, v_{\mathrm{sgn}(t)}\rangle \mp i\varepsilon}\, e^{-\varepsilon|t|} \qquad H_{\kappa,\varepsilon}(t) \underset{(H)}{:=} W(h_{\kappa,\varepsilon}(t))\, H_\varepsilon(t)\, W^\dagger(h_{\kappa,\varepsilon}(t))$$

$$\Omega_{\kappa,\varepsilon}(t,0) \overset{(U)}{:=} U_{\kappa,\varepsilon}(0,t)\, U_0(t,0) \overset{(E)}{=} e^{-i\alpha_{\kappa,\varepsilon}(t,0)}\, e^{-i\Phi(F_{\kappa,\varepsilon}(t,0))}\, e^{-i\beta_{\kappa,\varepsilon}(t,0)}$$

Try $\lim\limits_{t\to\pm\infty} \Omega_{\kappa,\varepsilon}(t,0)$ (A)

- $F_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{L^2(\mathbb{R}^3)\text{-}\lim} F_{\kappa,\varepsilon}(t,0)$ exists
- $\alpha_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim}\ \alpha_{\kappa,\varepsilon}(t,0)$ exists
- $\beta_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim}\ \beta_{\kappa,\varepsilon}(t,0)$ exists

$\Rightarrow$ $\Omega_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{\text{strong-}\lim}\ \Omega_{\kappa,\varepsilon}(t,0)$ exists

$\Omega_{\kappa,\varepsilon}$ Convergence Boost + 0-Mode Cutting

$$h_{\kappa,\varepsilon}(t) := \hat{e}/\sqrt{2W}\ \mathbb{1}_{B_\kappa(0)}\ \frac{1}{\omega - \langle \cdot, v_{\mathrm{sgn}(t)}\rangle \mp i\varepsilon}\, e^{-\varepsilon|t|} \qquad H_{\kappa,\varepsilon}(t) \overset{(H)}{:=} W(h_{\kappa,\varepsilon}(t))\, H_\varepsilon(t)\, W^{+}(h_{\kappa,\varepsilon}(t))$$

$$\Omega_{\kappa,\varepsilon}(t,0) \overset{(U)}{:=} U_{\kappa,\varepsilon}(0,t)\, U_0(t,0) \overset{(E)}{=} e^{-i\alpha_{\kappa,\varepsilon}(t,0)}\, e^{-i\Phi(F_{\kappa,\varepsilon}(t,0))}\, e^{-i\beta_{\kappa,\varepsilon}(t,0)}$$

Try $\lim\limits_{t\to\pm\infty} \Omega_{\kappa,\varepsilon}(t,0)$ (A)

- $F_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{L^2(\mathbb{R}^3)\text{-}\lim}\ F_{\kappa,\varepsilon}(t,0)$ exists
- $\alpha_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim}\ \alpha_{\kappa,\varepsilon}(t,0)$ exists
- $\beta_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim}\ \beta_{\kappa,\varepsilon}(t,0)$ exists

$\Omega_{\kappa,\varepsilon}(\pm\infty,0) := \underset{t\to\pm\infty}{\text{strong-}\lim}\ \Omega_{\kappa,\varepsilon}(t,0)$ exists

$\Rightarrow$ $\Omega_\kappa(\pm\infty,0) := \underset{\varepsilon\to 0}{\text{strong-}\lim}\ e^{i\alpha_{\kappa,\varepsilon}(\pm\infty,0)}\, e^{i\beta_{\kappa,\varepsilon}(\pm\infty,0)}\, \Omega_{\kappa,\varepsilon}(\pm\infty,0) = e^{-i\Phi(F_\kappa(\pm\infty,0))}$

Ω_{as} Comparison Dynamics

$$f_{as}(t) = \frac{(2\pi)^{-3/2}}{\sqrt{2\omega_k}} \left((2\pi)^{3/2} \hat{\varrho}^*(k) \right) e^{-i\langle k, x_{sgn(t)} + v_{sgn(t)} t \rangle}, \quad H_{as}(t) \underset{(H)}{:=} d\Gamma(\omega) + \Phi(f_{as}(t))$$

$$\Omega_{as}(t,0) \overset{(U)}{:=} U(0,t)\,U_{as}(t,0) \overset{(E)}{=} e^{-i\alpha_{as}(t,0)}\, e^{-i\Phi(F_{as}(t,0))}$$

Try $\lim\limits_{t\to\pm\infty} \Omega_\varepsilon(t,0)$ (A)

- $F_{as}(\pm\infty,0) := \underset{t\to\pm\infty}{L^2(\mathbb{R}^3)\text{-}\lim}\, F_{as}(t,0)$ exists
- $\alpha_{as}(\pm\infty,0) := \underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim}\, \alpha_{as}(t,0)$ exists

Ω_{as} Comparison Dynamics

$$f_{as}(t) = \frac{(2\pi)^{-3/2}}{\sqrt{2\omega_k}} \left((2\pi)^{3/2} \hat{g}^*(k) \right) e^{-i\langle k, x_{sgn(t)} + v_{sgn(t)} t \rangle} \, , \quad H_{as}(t) \overset{(H)}{:=} d\Gamma(\omega) + \Phi(f_{as}(t))$$

$$\Omega_{as}(t,0) \overset{(U)}{:=} U(0,t)U_{as}(t,0) \overset{(E)}{=} e^{-i\alpha_{as}(t,0)} e^{-i\Phi(F_{as}(t,0))}$$

Try $\lim\limits_{t\to\pm\infty} \Omega_\varepsilon(t,0)$ (A)

- $F_{as}(\pm\infty,0) := \underset{t\to\pm\infty}{L^2(\mathbb{R}^3)\text{-}\lim} F_{as}(t,0)$ exists
- $\alpha_{as}(\pm\infty,0) := \underset{t\to\pm\infty}{\mathbb{C}\text{-}\lim} \alpha_{as}(t,0)$ exists

$$\Rightarrow \Omega_{as}(\pm\infty,0) := e^{i\alpha_{as}(\pm\infty,0)} \underset{t\to\pm\infty}{\text{strong-}\lim} \Omega_{as}(t,0)$$

$$= e^{-i\Phi(F_{as}(\pm\infty,0))}$$

Scattering Probabilities

$\# = \phi, K, as$
$\quad {}^{\uparrow} m>0$

$$\Omega_{\#}(\pm\infty, 0) = e^{-i\Phi(F_{\#}(\pm\infty,0))}$$

Regard $u_1, \dots, u_n, v_1, \dots, v_m \in C_c^{\infty}(\mathbb{R}^3, \mathbb{C})$

s.t. $\operatorname{supp}(u_i) \cap \operatorname{supp}(v_j) = \emptyset$

$$\mathbb{P}_{\#}(u_1, \dots, u_n \to v_1, \dots, v_m)$$
$$:= \left| \langle 0 | a(u_1) \dots a(u_n) \, \Omega^{+}(-\infty, 0) \, \Omega(+\infty, 0) \, a^{+}(v_1) \dots a^{+}(v_m) | 0 \rangle \right|^2$$

Scattering Probabilities

$\# = \emptyset,\ K,\ as$ ($m>0$)

$$\Omega_{\#}(\pm\infty,0) = e^{-i\Phi(F_{\#}(\pm\infty,0))}$$

Regard $u_1,\dots,u_n,\ v_1,\dots,v_m \in C_c^{\infty}(\mathbb{R}^3,\mathbb{C})$

s.t. $\mathrm{supp}(u_i) \cap \mathrm{supp}(v_j) = \emptyset$

$$\mathbb{P}_{\#}(u_1,\dots,u_n \to v_1,\dots,v_m)$$

$$:= \left|\langle 0 | a(u_1)\dots a(u_n)\, \Omega^{+}(-\infty,0)\, \Omega(+\infty,0)\, a^{+}(v_1)\dots a^{+}(v_n) | 0\rangle\right|^2$$

$$= \mathbb{P}_{\#}(\emptyset \to \emptyset) \left(\prod_{i=1}^{n} |\langle u_i, \Delta F_{\#}(\pm\infty,0)\rangle|^2\right)\left(\prod_{j=1}^{m} |\langle \Delta F_{\#}(\pm\infty,0), v_j\rangle|^2\right)$$

$$\Delta F_{\#}(\pm\infty,0) := F_{\#}(-\infty,0) - F_{\#}(+\infty,0) \ \text{"} \overset{(\#=\emptyset)}{=} i\sqrt{2W}\,\hat{e} \ \text{"} \ \frac{1}{2}\text{"}\ \mathcal{F}(\varphi_\delta^{ret} - \varphi_\delta^{adv})\text{"}$$

Scattering Probabilities

Regard $k_1, \dots, k_n, p_1, \dots, p_m \in \mathbb{R}^3 \setminus \{0\}$ s.t. $k_i \neq p_j$

$$\mathbb{P}_{\#}(k_1, \dots, k_n \to p_1, \dots, p_m) := \mathbb{P}_{\#}(\emptyset \to \emptyset) \left(\prod_{i=1}^{n} |\Delta F_{\#}(\pm\infty, 0)(k_i)|^2 \right) \left(\prod_{j=1}^{m} |\Delta F_{\#}(\pm\infty, 0)(p_j)|^2 \right)$$

$\# = \emptyset$: Detector Threshold E :

$K_1, \dots, K_n, P_1, \dots, P_m$ s.t. $\omega_{K_i}, \omega_{P_j} > E$

Scattering Probabilities

Regard $k_1,\dots,k_n,p_1,\dots,p_m\in\mathbb{R}^3\setminus\{0\}$ **s.t.** $k_i\neq p_j$

$$\mathbb{P}_\#(k_1,\dots,k_n\to p_1,\dots,p_m) := \mathbb{P}_\#(\emptyset\to\emptyset)\left(\prod_{i=1}^{n}|\Delta F_\#(\pm\infty,0)(k_i)|^2\right)\left(\prod_{j=1}^{m}|\Delta F_\#(\pm\infty,0)(p_j)|^2\right)$$

$\# = \emptyset$: Detector Threshold E :

$K_1,\dots,K_n,P_1,\dots,P_m$ s.t. $\omega_{K_i},\omega_{P_j}>E$

$$\begin{aligned}&\mathbb{P}(K_1,\dots,K_n\to P_1,\dots,P_m+\text{soft photons})\\ &=\sum_{N=0}^{\infty}\frac{1}{N!}\left(\prod_{j=1}^{N}\int d^3p_j\,\Theta(E-\omega_{p_j})\right)\mathbb{P}(K_1,\dots,K_n\to P_1,\dots,P_m,p_1,\dots,p_N)\\ &=e^{-\|\underbrace{(1-\Theta(E-\omega))}_{\text{IR-finite}}\Delta F(\pm\infty,0)\|^2_{L^2}}\left(\prod_{i=1}^{n}|\Delta F_\#(\pm\infty,0)(K_i)|^2\right)\left(\prod_{j=1}^{m}|\Delta F_\#(\pm\infty,0)(P_j)|^2\right)\end{aligned}$$

Scattering Frequences

Regard $K_1, \dots, K_n, P_1, \dots, P_m \in \mathbb{R}^3 \setminus \{0\}$ s.t. $K_i \neq P_j$
and $\# = \emptyset, K, as$

$$f_\#(K_1, \dots, K_n \to P_1, \dots, P_m ; \text{baseline}) := \frac{\mathbb{P}_\#(K_1, \dots, K_n \to P_1, \dots, P_m)}{\mathbb{P}_\#(\text{baseline})}$$

IR + UV
finite

Scattering Frequences

Regard $K_1, \dots, K_n, P_1, \dots, P_m \in \mathbb{R}^3 \setminus \{0\}$ s.t. $K_i \neq P_j$

and $\# = \emptyset, \kappa, as$

$$f_\#(K_1, \dots, K_n \to P_1, \dots, P_m ; \text{baseline}) := \frac{\mathbb{P}_\#(K_1, \dots, K_n \to P_1, \dots, P_m)}{\mathbb{P}_\#(\text{baseline})}$$

IR + UV finite

and for $m = 0$,

$\|K_1\|, \dots, \|K_n\|, \|P_1\|, \dots, \|P_m\| > \kappa$

$$f(K_1, \dots, K_n \to P_1, \dots, P_m ; \text{baseline})$$

$$= f_\kappa(K_1, \dots, K_n \to P_1, \dots, P_m ; \text{baseline})$$

$$\neq f_{as}(K_1, \dots, K_n \to P_1, \dots, P_m ; \text{baseline})$$