Entanglement and separable states in relativistic QFT

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Colloquium on Mathematics and Foundations of Quantum Theory,
Poznań, 8 April 2025

KS, J. Phys. A: Math. Theor. **56** 505201 (2023)
Hollands-KS, Springer Briefs in Mathematical Physics Vol. 34 (2018)

Heisenberg-Programm
DFG Deutsche Forschungsgemeinschaft

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A brief history of entanglement

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1930-1935: Entanglement is discovered. (Schrödinger, Einstein)

1935: The EPR paradox uses entanglement to argue that QM is

a. incomplete: there must be hidden variables, or
b. non-local: correlations between measurement outcomes at spacelike separation without prior common causes.

Option b was unacceptable to Einstein.

1964: Bell derives the Bell inequality, which is

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2. sometimes violated by quantum mechanics.

Option 2+EPR entails that nature (and QM) is non-local.

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What nature has to say

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$\rightarrow$ Nature is non-local!*

*To keep track of the localisation of observables we use QFTs.

1961: The Reeh-Schlieder Theorem indicates that vacuum states of Wightman QFTs are entangled between all regions of space.

$\rightarrow$ Entanglement is the rule, rather that an exception.

What to do about non-locality?

1. Standard QM and QFT:
   Non-locality enters through measurement outcomes, but there is no communication faster than light. (Causal structure not violated by quantum experiment.)
   $\rightarrow$ Some uneasiness about the lack of realistic interpretations and the special rôle of measurements.

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1. Standard QM and QFT:
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2. Bohmian mechanics:
   Non-local dynamics in spacetime, completing QM by adding particle positions (but not momenta) as hidden variables.
   $\rightarrow$ Difficult to extend to relativistic QFT.

What to do about non-locality?

3. Spacetime is emergent:
   Pointlike localisation is not fundamental, but an effective property that emerges only at low energy scales.
   $\rightarrow$ No full QGR theory available to prove emergence.

   Note in support:
   In QFTs in vacuum the entanglement entropy decays with distance. (Non-locality is harder to see at long distances/low energy.)

This talk focuses on entanglement in QFT.

Entanglement entropy in QM and QFT

Entanglement in QM

Let ρ a density matrix on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ (i.e. $\rho \geq 0$, $\mathrm{tr}_{\mathcal{H}}\rho = 1$).

- ρ is a product state iff $\rho = \rho_A \otimes \rho_B$ iff for all $a \in \mathcal{B}(\mathcal{H}_A)$, $b \in \mathcal{B}(\mathcal{H}_B)$

$$\mathrm{tr}_{\mathcal{H}}\rho(a \otimes b) = \mathrm{tr}_{\mathcal{H}}\rho(a \otimes I) \cdot \mathrm{tr}_{\mathcal{H}}\rho(I \otimes b)\,.$$

  $\rightarrow$ No measurement correlations between A and B.

- ρ is a separable state iff ρ is a mixture of product states,

$$\rho = \sum_n \lambda_n \rho_A^{(n)} \otimes \rho_B^{(n)}$$

  with $\lambda_n \geq 0$, $\sum_n \lambda_n = 1$.
  $\rightarrow$ Only classical measurement correlations between A and B.

- ρ is entangled iff it is not separable.

Entanglement entropy

We want a kind of "distance" from a state to the set of separable states.

The entanglement entropy S_{EE} is defined in terms of the relative entropy:

$$S_{\mathrm{EE}}(\rho) = \inf_{\rho' \text{ separable}} S_{\mathrm{rel}}(\rho, \rho') ,$$

$$S_{\mathrm{rel}}(\rho, \rho') = \mathrm{tr}\rho(\log \rho - \log \rho') .$$

For the relative entropy we have:

- $S_{\mathrm{rel}}(\rho, \rho') = \mathrm{tr}\rho(\log \rho - \log \rho') \geq 0$ with $= 0$ iff $\rho = \rho'$,
- $S_{\mathrm{rel}}(\rho, \rho') \neq S_{\mathrm{rel}}(\rho', \rho)$ in general,
- no triangle inequality in general.

$S_{\mathrm{rel}}(\rho, \rho')$ is the expected amount of information gained when we learn that the state is ρ and we previously thought it was ρ'. (Baez-Fritz, 2014)

Important bad entanglement measures

- The von Neumann entropy: for $\rho_A = \mathrm{tr}_{\mathcal{H}_B}\rho$,

$$S_{\mathrm{vN}}(\rho_A) = -\mathrm{tr}\rho_A \log(\rho_A) \geq 0.$$

- The violation of the Bell (CHSH) inequality:

$$\beta(\rho) = \frac{1}{2} \sup_{a_i, b_i} \mathrm{tr}\rho(a_1 \otimes (b_1 + b_2) + a_2 \otimes (b_1 - b_2)) \in [1, \sqrt{2}]$$

with $a_i^* = a_i$, $b_i^* = b_i$, $\|a_i\|, \|b_i\| \leq 1$. (Summers-Werner,1995)

For pure states $\rho = |\psi><\psi|$ with $\psi \in \mathcal{H}$:

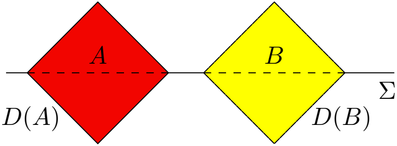
$$\rho \text{ is entangled} \quad \Leftrightarrow \quad S_{\mathrm{vN}}(\rho) > 0 \quad \Leftrightarrow \quad \beta(\rho) > 1.$$

Problems with mixed states:

1. $S_{\mathrm{vN}}(\rho_A) > 0$ for separable mixed states (not specific enough),
2. $\beta(\rho) = 1$ for some entangled states (not sensitive enough). (Gisin, 1996)

Entanglement in QFT

Let $A, B \subset \Sigma$ disjoint, bounded, open regions in a Cauchy surface.



$D(A)$, $D(B)$ domains of dependence $\mathcal{A}_A, \mathcal{A}_B$, $\mathcal{A}_{A\cup B}$ associated algebras of observables.

- If $A \cap B = \emptyset$, then $[\mathcal{A}_A, \mathcal{A}_B] = \{0\}$ (Einstein causality)
- Typically we have a unitary $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ with (split property)

$$U(\mathcal{A}_{A\cup B})U^* = \mathcal{A}_A \otimes \mathcal{A}_B$$

  if and only if $\overline{A} \cap \overline{B} = \emptyset$. (Doplicher-Longo, 1984)
- If $A \cup B \neq \Sigma$, any reasonable state restricted to $\mathcal{A}_{A\cup B}$ is mixed.
- $\mathcal{A}_A \neq \mathcal{B}(\mathcal{H})$, but S_{rel} and S_{EE} can still be defined using Tomita-Takesaki modular theory.

Results on entanglement entropy in QFT

Consider a static spacetime. For fixed A and varying $r = \mathrm{dist}(A, B)$,

$$S_{\mathrm{EE}}(\Omega) \lesssim C\, e^{-\frac{m}{2} r}, \qquad mr \gg 1$$

for the ground state Ω of massive ($m > 0$)

- free minimally coupled scalar fields,
- free Majorana fields,
- integrable QFTs on $(1+1)$-dimensional Minkowski space with a factorizing S-matrix (e.g. sinh-Gordon).

(Hollands-KS, 2018)

In other examples (including $m = 0$) the decay is at least polynomial.

Question: Are the separable states that yield these estimates physically reasonable? (Symmetric? Hadamard?)

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Separable states in relativistic QFT

Toy model: free scalar QFT

Let ϕ be a free scalar QFT in a globally hyperbolic spacetime M.
A quasi-free Hadamard state ω is determined by its 2-point distribution

$$\omega_2(f,g) = \langle \Omega, \phi(f)\phi(g)\Omega \rangle_\omega$$

which satisfies for $K = -\Box + m^2 + \xi R$

1. $\omega_2(\bar{f}, f) \geq 0$ for all $f \in C_0^\infty(M)$ (positive type)
2. $K_x \omega_2(x,y) = K_y \omega_2(x,y) = 0$ weakly (EOM)
3. $\omega_2(x,y) - \omega_2(y,x) = iE(x,y)$ a fixed distribution (CCR)
4. $WF(\omega_2) \subset \mathcal{C}$ a fixed set (Hadamard condition)

ω is a product state between $D(A), D(B) \subset M$,

$$\omega(e^{i\phi(f)} e^{i\phi(g)}) = \omega(e^{i\phi(f)})\omega(e^{i\phi(g)}) \quad \forall f \in C_0^\infty(D(A)), g \in C_0^\infty(D(B))$$

iff ω_2 vanishes on $D(A) \times D(B)$ and (hence) $D(B) \times D(A)$.

Initial data formulation

Let Σ be a smooth spacelike Cauchy surface in M. To handle the EOM we will use initial data for ω_2 on Σ of the form

$$\left(\omega_{2,ij}\right)_{i,j=0,1} = \begin{pmatrix} \omega_{2,00}|_{\Sigma\times\Sigma} & \partial_{y^0}\omega_{2,00}|_{\Sigma\times\Sigma} \\ \partial_{x^0}\omega_{2,00}|_{\Sigma\times\Sigma} & \partial_{x^0}\partial_{y^0}\omega_{2,00}|_{\Sigma\times\Sigma} \end{pmatrix} .$$

Given any ω_2, we find other Hadamard 2-point distributions as

$$\omega_2'(x,y) = \omega_2(x,y) + w_2(x,y)$$

with $w_2 \in C^\infty(M^2)$, s.t. $w_2(y,x) = w_2(x,y)$, $K_x w_2(x,y) = 0$ and

$$w_2(\bar{f},f) \geq -\omega_2(\bar{f},f) \quad \forall f \in C_0^\infty(M) .$$

Clearly ≥ 0 is sufficient.

A simple construction

Given any ω_2 and bounded open regions $A, B \subset \Sigma$ with $\overline{A} \cap \overline{B} = \emptyset$, we want to remove the correlations between $D(A)$ and $D(B)$ in M.

Let $\chi_A, \chi_B \in C_0^\infty(\Sigma)$ such that

$$\chi_{A/B} \equiv 1 \text{ on } A/B \quad \mathrm{supp}(\chi_A) \cap \mathrm{supp}(\chi_B) = \emptyset \,.$$

Consider $v_2 \in C_0^\infty(\Sigma^2, \mathrm{Mat}(2, \mathbb{R}))$ defined by

$$\left(v_{2,ij}(x,y)\right) := -\chi_A(x)\chi_B(y) \cdot \left(\omega_{2,ij}(x,y)\right) \,.$$

Warning: $w_{2,ij}(x,y) = v_{2,ij}(x,y) + v_{2,ji}(y,x)$ removes the correlations, but destroys positive type!

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v_2 is the integral kernel of a compact operator L on $L^2(\Sigma, \mathbb{C}^2)$. We can write $L = U|L|$ with $|L| = \sqrt{L^* L}$ and U a partial isometry. Then

$$\tilde{L} := L + L^* + |L| + |L^*| \geq 0 .$$

Proposition

$\tilde{L}$ has a smooth integral kernel $\tilde{v}_2 \in C_0^\infty(\Sigma^2, \mathrm{Mat}(2, \mathbb{R}))$, such that

- *$\tilde{v}_2$ is of positive type*
- *$\tilde{v}_2 = -\left(\omega_{2,ij}\right)$ on $A \times B$ and $B \times A$,*

Taking $(w_{2,ij}) = \tilde{v}_2$ yields a Hadamard two-point distribution $\omega_2' = \omega_2 + w_2$ for a product state between $D(A)$ and $D(B)$.

$S_{\mathrm{rel}}(\omega, \omega')$ is still under investigation.

Symmetric separable states

Symmetric separable states

Consider $3+1$-dimensional Minkowski space and assume $m > 0$. We write $x = (x_0, \mathbf{x})$.

Theorem

Given any $R > 0$, there exist quasi-free Hadamard states ω which are stationary, homogeneous, isotropic and s.t.

1. $\omega_2(x, x') = 0$ *if* $(x, x') \in S = \{\|\mathbf{x} - \mathbf{x}'\| > R + |x_0 - x_0'|\}$,
2. $\omega(T_{00}^{\mathrm{ren}}(x)) \leq 10^{31} m^4 \frac{e^{-\frac{1}{4}mR}}{(mR)^8}$.

- (1) shows that ω is a product state between any regions spacelike separated by a distance $> R$.
- $\omega(T_{00}^{\mathrm{ren}}(x))$ is constant and cannot be arbitrarily small.
- The upper bound in (2) is not sharp, but falls off fast as R grows.

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Construction

We use the Minkowski vacuum $\omega^{(0)}$ as reference state,

$$\omega_2(x - x') = \omega_2^{(0)}(x - x') + w(x - x'),$$

where w is smooth, real, even with $Kw = 0$ and $\hat{w} \geq 0$.

At $x^0 = 0$ we have $\partial_0 w|_{x^0=0} \equiv 0$. We want $w_0(\mathbf{x}) = w(0, \mathbf{x})$ with

1. $\widehat{w_0} \geq 0$ (positive type)
   By Bochner's Theorem $w(x - x')$ is of positive type iff $\hat{w} \geq 0$.
2. $w_0(\mathbf{x}) = -\omega_{2,00}^{(0)}(0, \mathbf{x})$ if $\|\mathbf{x}\| \geq 0$ (separability)
3. $\omega(T_{00}^{\mathrm{ren}}(0)) = (-\Delta + m^2)w_0(0)$ small (energy density)

If $\chi \in C_0^\infty((-R, R), \mathbb{R})$ with $\chi \equiv 1$ near 0 we can try

$$w_0(\mathbf{x}) = (\chi(\|\mathbf{x}\|) - 1)\omega_{2,00}^{(0)}(0, \mathbf{x}).$$

This removes correlations, but destroys the positive type.

Test functions of positive type

Let $l > 0$ and $\epsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ monotonically decreasing.

Theorem (Ingham, 1934)

There exists non-zero $f \in C_0^\infty((-l,l),\mathbb{R})$ such that $|\hat{f}(k)| \leq e^{-|k|\epsilon(|k|)}$ for all $k \in \mathbb{R}$ iff $\int_1^\infty \frac{\epsilon(k)}{k}\mathrm{d}k < \infty$.

To omplete our construction, we want to control lower bounds on $\hat{f}$.

Theorem (KS, 2023)

Let also $\gamma \in (0,1)$. If $\int_1^\infty \frac{\epsilon(k)}{k}\mathrm{d}k < \infty$ and $\liminf_{k\to\infty} k^\gamma \epsilon(k) > 0$, there exists $g \in C_0^\infty((-l,l),\mathbb{R}_{\geq 0})$ such that $\int g = 1$ and $\hat{g}(k) \geq e^{-|k|\epsilon(|k|)}$.

- The proof constructs g with repeated convolutions.
- This theorem is in one direction only and may not be sharp.

Comments on the proof:

A standard construction of test-functions in $C_0^\infty([0,a],\mathbb{R})$ takes

$$f(x) = \lim_{n\to\infty} \frac{1}{a_1} 1_{[0,a_1]} * \frac{1}{a_2} 1_{[0,a_2]} * \ldots * \frac{1}{a_n} 1_{[0,a_n]}$$

where $a_1 \geq a_2 \geq \ldots > 0$ with $\sum_{n=1}^\infty a_n < \infty$. We replace $\frac{1}{a_n} 1_{[0,a_n]}$ by

$$\eta_{a_n}(x) = \frac{3}{2}\frac{1}{a_n}\left(1_{[-\frac{1}{2},\frac{1}{2}]} * 1_{[-\frac{1}{2},\frac{1}{2}]}\right)^2 \left(\frac{x}{a_n}\right).$$

We can then construct $g \in C_0^\infty([-a,a],\mathbb{R})$ of positive type and

- estimate $\mathrm{supp}(g)$, $\|g^{(k)}\|_1$ and $\|g^{(k)}\|_\infty$ in terms of the a_n,
- find suitable a_n to prove the theorem.

Notes:

1. Any Gevrey class function can be dominated by a test-function of positive type with arbitrarily small support near 0.
2. The only prior result I know of controlling lower bounds is asymptotic. (Fewster-Ford, 2015)

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Conclusions and outlook

Conclusions and open questions

- Entanglement shows that nature is non-local (in some sense).
- We use QFTs to investigate entanglement $\leftrightarrow$ localisation.
- Nice separable states exist for free scalar QFTs:
  - as small perturbations of a given quasi-free state,
  - as translation invariant states in Minkowski space, separable over distances $\geq R$ at a fixed time.
- Energy bounds suggest: separability is easier/cheaper to maintain at large R.
  $\rightarrow$ Can we get sharper bounds on the energy (density)?
- Entanglement entropy in vacuum falls off with distance.
  $\rightarrow$ Can we get the same decay using nice separable states?