Baryogenesis in conformally flat spacetimes

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Introduction

- **Guiding question:** Where does the matter/antimatter asymmetry in the Universe come from?
- *Baryogenesis*: Matter creation process responsible for the observed asymmetry.
  $\rightarrow$Can it be described in the language of global analysis? What triggered it?
- Our assumptions (motivated from CFS):
  (i) **On the regularity of spacetime:** On the very small scales, below a minimum length $\varepsilon > 0$ (the regularization length), the manifold structure breaks down.
  $\rightarrow$ Consequence: Start from distributional sections of *SM* (e.g., weak solutions to the Dirac equation) and regularize them.

(ii) **On the physical input:** The regularization operators

$$R_\varepsilon : H^{1,2}_{\mathrm{loc}}(M, SM) \to C^0(M, SM)$$

have an important physical significance and depend on the spacetime point: for any $p \in M$, $R_\varepsilon(p) : H^{1,2}_{\mathrm{loc}}(M, SM) \to S_pM$.
Moreover, the dynamics of R_ε:

$$M \to \mathcal{L}(H^{1,2}_{\mathrm{loc}}(M, SM), SM) \,, \quad p \mapsto R_\varepsilon(p)$$

is determined by the spinor dynamics.
→Consequence: Shift attention from distributional spinors to regularization operators and their dynamics.

(iii) **On the dynamics of the spinors:** Aim to describe spinors which evolve according to a slight modification of Dirac dynamics.

Introduction

- **Ansatz:** Introduce a certain dynamics for R_ε (described by a timelike vector field $u : M \to TM$; CFS input) which yields a spinor dynamics deviating slightly from the Dirac dynamics.
- Schematically:

$$\text{Spinor dynamics (Dirac)} \to \text{Dynamics of } R_\varepsilon \to \text{No baryogenesis}$$

$$\text{Dynamics of } R_\varepsilon \text{ (CFS input)} \to \text{Spinor dynamics} \to \text{Baryogenesis?}$$

The setup: Preliminaries

- In $(\mathbb{R}^4, \eta)$ the Dirac Hamiltonian H_η is a selfadjoint operator with absolutely continuous spectrum $\sigma(H_\eta) = (-\infty, -m] \cup [m, \infty)$.
  $\rightarrow$ *Particles* (resp. *antiparticles*) are eigenstates associated to positive (resp. negative) eigenvalues of H_η.
- **Idea:** In a globally hyperbolic (M, g), describe baryogenesis as a relative change of spectral subspaces of a *generalization* of H_η.
- **Problems with H_g:**
  - H_g is not symmetric unless (M, g) is stationary.
  - H_g describes exclusively Dirac dynamics of spinors.

The setup: Preliminaries

Geometric setup:

- From now on, consider conformally flat spacetimes $(\mathbb{R}^4, g)$, i.e.

$$g = \Omega^2 \eta = \Omega^2(t, r, \theta, \varphi)(dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin\theta^2 d\varphi^2)$$

  with $\Omega : M \to (0, \infty)$ smooth and the foliation $(N_t)_{t\in\mathbb{R}}$ given by the level sets of the global time function t.

- Moreover, we denote

$$\mathcal{H}_{t,g} := L^2(N_t, SM) ,$$

  with a suitable spinor scalar product $(\cdot|\cdot)_t$.

- Finally, *the regularizing vector field* $u : M \to TM$ is a timelike and future directed vector field describing the regularization dynamics.

Definition:

(i) The *symmetrized Hamiltonian* A_t with $D(A_t) = C_0^\infty(N_t, SM)$,

$$A_t := \frac{1}{4}\{u^0, H_g + H_g^*\} + \frac{i}{4}\{u^\mu, \nabla^s_\mu - (\nabla^s_\mu)^*\} \ ,$$

is an essentially self-adjoint operator (cf. [4, Lem. 5.3], proof relies on Chernoff's criteria, see [1]).
$\rightarrow$ From now on, denote with the same symbol the self-adjoint extension $A_t : \mathcal{D} \supset C_0^\infty(N_t, SM) \rightarrow \mathcal{H}_{t,g}$.

Remark:

- Given the selfadjoint operator $A_t : \mathcal{D} \supset C_0^\infty(N_t, SM) \to \mathcal{H}_{t,g}$, $\chi_I(A_t) : \mathcal{D} \to \mathcal{H}_{t,g}$ is a densely defined bounded operator ($I \subset \mathbb{R}$). Hence, there exists a unique extension

$$\chi_I(A_t) : \mathcal{H}_{t,g} \to \mathcal{H}_{t,g}$$

- If (M, g) has a bounded geometry and I is a bounded interval, $\chi_I(A_t)$ is a regularization operator (cf. [5, Proposition 2.7], proof relies on elliptic regularity and Sobolev embedding theorem):

$$\chi_I(A_t) : \mathcal{H}_{t,g} \to C^\infty(N_t, SM)$$

  Its dynamics is determined by the dynamics of $u : M \to TM$.

The setup: Main concepts

(ii) The *locally rigid operator* $V_{t_0}^t : \mathcal{H}_{t_0}^\varepsilon \to \mathcal{H}_{t,g}$ is

$$V_{t_0}^t := \lim_{k_{\max}\to\infty} \chi_I(A_t) U_{t-\Delta t}^t \cdots \chi_I(A_{t_0+\Delta t}) U_{t_0}^{t_0+\Delta t} \quad \text{with} \quad \Delta t := \frac{t-t_0}{k_{\max}}.$$

where $\mathcal{H}_{t_0}^\varepsilon := \chi_I(A_{t_0})(\mathcal{H}_{t_0,g}) \subset \mathcal{H}_{t_0,g}$, $I := (-1/\varepsilon, -m)$ and for any $t_k < t_{k+1}$, $U_{t_k}^{t_{k+1}} : \mathcal{H}_{t_k,g} \to \mathcal{H}_{t_{k+1},g}$ is the unitary Dirac evolution operator.

Moreover, set $\mathcal{H}_t^\varepsilon := V_{t_0}^t(\mathcal{H}_{t_0}^\varepsilon)$.

The setup: Main concepts

(iii) The *rate of baryogenesis*:

$$B_t := \frac{d}{dt}\mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}}(\eta(\tilde{H}_\eta)\chi_I(A_t))$$

with $I := (-1/\varepsilon, -m)$ and $\tilde{H}_\eta := \tilde{U}^{-1}H_\eta\tilde{U}$, where $\tilde{U} : \mathcal{H}_{t,g} \to \mathcal{H}_{t,\eta}$ is a unitary operator. Moreover, $\eta \in C_0^\infty((-\Lambda, \Lambda), [0, \infty))$ is a smooth cutoff function with $m \ll \Lambda \ll \frac{1}{\varepsilon}$.

In a nutshell:

Dynamics of R_ε (CFS input) $\to$	Spinor dynamics $\to$	Baryogenesis?
$(u : M \to TM)$	$(V_{t_0}^t : \mathcal{H}_{t_0}^\varepsilon \to \mathcal{H}_{t,g})$	(Next slides...)

Quantifying the rate of baryogenesis

Goal: Quantify B_t perturbatively for conformally flat spacetimes.

Assumptions:

(i) $(\mathbb{R}^4, g)$ has a bounded geometry.

(ii) A_t has an absolutely continuous spectrum.

(iii) $A_t = \tilde{H}_\eta + \Delta A$, where $\Delta A : C^\infty(N_t, SM) \to C_0^\infty(N_t, SM)$ has smooth compactly supported coefficients and satisfies that for any $\omega \in \rho(\tilde{H}_\eta)$

$$||R_\omega(\tilde{H}_\eta)\Delta A|| < 1$$

These assumptions allow for a perturbative study of B_t.

Quantifying the rate of baryogenesis

Step 1: Perturb the spectral projection operator.

$$R_\omega(A_t) = (1 + R_\omega(\tilde{H}_\eta)\Delta A(t))^{-1} R_\omega(\tilde{H}_\eta) = \sum_{p=0}^{\infty} (-R_\omega(\tilde{H}_\eta)\Delta A(t))^p R_\omega(\tilde{H}_\eta)$$

$$=: \sum_{p=0}^{\infty} R_\omega^{(p)}(A_t)$$

By absolute continuity of the spectrum of A_t

$$\chi_I(A_t) = \int_I F_{\omega'}(A_t) d\omega' = \frac{1}{2\pi i} \underset{\delta \to 0^+}{\text{s-lim}} \int_I R_{\omega' + is\delta}(A_t)\Big|_{s=-1}^{s=1} d\omega' \,,$$

where $F_\omega(A_t) : \mathcal{H}_{t,g} \to \mathcal{H}_{t,g}$ and we used Stone's formula. Analogous for $\tilde{H}_\eta$.

Quantifying the rate of baryogenesis

Step 2: Obtain the perturbative power expansion of B_t.
We start with the following operator product

$$\eta_\Lambda(\tilde{H}_\eta)\chi_I(A_t) = \int_{-\infty}^{\infty} d\omega' \eta_\Lambda(\omega') F_{\omega'}(\tilde{H}_\eta) \int_{-\frac{1}{\varepsilon}}^{-m} d\omega F_\omega(A_t)$$

$$= \frac{1}{2\pi i} \operatorname*{s-lim}_{\delta\to 0^+} \int_{-\frac{1}{\varepsilon}}^{-m} d\omega \int_{-\infty}^{\infty} d\omega' \eta_\Lambda(\omega') F_{\omega'}(\tilde{H}_\eta) R_{\omega+is\delta}(A_t)\Big|_{s=-1}^{s=1}$$

Thus, B_t is

$$B_t := \frac{d}{dt} \mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}}(\eta(\tilde{H}_\eta)\chi_I(A_t))$$

$$= \frac{d}{dt}\left(\frac{1}{2\pi i} \operatorname*{s-lim}_{\delta\to 0^+} \int_{-\frac{1}{\varepsilon}}^{-m} d\omega \int_{-\infty}^{\infty} d\omega' \eta_\Lambda(\omega')\, \mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}}\left(F_{\omega'}(\tilde{H}_\eta) R^{(p)}_{\omega+is\delta}(A_t)\right)\Big|_{s=-1}^{s=1}\right)$$

$$=: \sum_{p=0}^{\infty} B_t^{(p)}$$

where the operator product is trace-class ([4, Lemma 7.4]).

Step 3: Determine the strong limit $\delta \to 0^+$ (cf. [4, Lemma 7.3]).

For $p = 2$ we obtain

$$B_t^{(2)} = -\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \, \partial_\omega \left(\eta_\Lambda(\omega) \, \mathrm{tr}_{\overline{\mathcal{H}_t^\varepsilon}} \left(\frac{d}{dt} \tilde{Q}(\omega, \omega') \right) \frac{g(\omega') - g(\omega)}{\omega' - \omega} \right.$$

where $\tilde{Q}(\omega, \omega') := \Delta AF_\omega(\tilde{H}_\eta) \Delta AF_{\omega'}(\tilde{H}_\eta) : \mathcal{H}_{t,g} \to C_0^\infty(N_t, SM)$ and g is the characteristic function of $I := (-1/\varepsilon, -m)$. Note that $\tilde{Q}(\omega = -m, \omega' = -m) = 0$.

Step 4: Compute the trace in momentum space.

The computation yields (cf. [5, Theorem 1.1])

$$B_t^{(2)} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{4\omega_k \omega_{k'}} \frac{1}{(\omega_{k'} + \omega_k)^2} G_{\Omega,m,u}(\vec{k}, \vec{k'}) \, ,$$

where $\omega_k := \sqrt{|\vec{k}|^2 + m^2}$, $\omega_{k'} := \sqrt{|\vec{k'}|^2 + m^2}$ and $G_{\Omega,m,u} : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is a smooth function which depends on the conformal factor Ω, on the mass m and on the vector field u.

Quantifying the rate of baryogenesis

Example: Let $u = \partial_t$ and $m \neq 0$. Then,

$$\Delta A = (\Omega - 1) m \gamma_{\eta 0} \ : \ C^\infty(N_t, SM) \to C_0^\infty(N_t, SM)$$

and the second order rate of baryogenesis is

$$B_t^{(2)} = 2m^2 \int_0^\infty \frac{d\rho}{(2\pi)^4} \left(\hat{\alpha}_1'(\rho) \hat{\alpha}_2(-\rho) + \hat{\alpha}_2'(\rho) \hat{\alpha}_1(-\rho) \right) Q(\rho) \ ,$$

where $\alpha_1 = \frac{d\Omega}{dt}$, $\alpha_2 = (\Omega - 1)$, $\alpha_i' = \Omega^3 \alpha_i$, Q is smooth and $\rho := \frac{1}{2}|\vec{k} + \vec{k'}|$.

Quantifying the rate of baryogenesis

Remarks on B_t (cf. [2], [4], [5]):

(i) No baryogenesis if spinors follow Dirac dynamics.

(ii) No baryogenesis if $m = 0$ and $u = \partial_t (\implies \Delta A = 0)$.

(iii) $B_t^{(0)} = B_t^{(1)} = 0$, but $B_t^{(2)} \neq 0$ in general (for $m \neq 0$ and/or $u \neq \partial_t$).

(iv) What about Minkowski spacetime?

$$B_t^\eta = 0 \quad \text{if} \quad u = \partial_t .$$

However, for general conformally flat spacetime, $B_t \propto m^2$ when $u = \partial_t$.

Discussion

- **Outlook:**
  (i) Look at concrete cosmological spacetimes
  $\rightarrow$ *Enough* baryogenesis in order to describe observed matter/antimatter asymmetry?
  (ii) Consider inflationary spacetimes with metric perturbations which break homogeneity and isotropy.

Thank you for your attention!

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