Electrons, Photons, and Darwin's Dream

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20th Colloquium on
Mathematics and Foundations of Quantum Theory
Adam Mickiewicz University, Poznań
08.04.2025

Abstract

"The Compton effect, at its discovery, was regarded as a simple collision of two bodies, and yet the detailed discussion at the present time involves the idea of the annihilation of one photon and the simultaneous creation of one among an infinity of other possible ones. We would like to be able to treat the effect as a two-body problem, with the scattered photon regarded as the same individual as the incident, in just the way we treat the collisions of electrons."

-**C. G. Darwin,** "Notes on the Theory of Radiation" (1932).

Relativistic Quantum Mechanics of N particles

- Wavefunction of N relativistic spin-1/2 particles:
  $\Psi = \Psi(x_1, \dots, x_N) \in \left(\mathbb{C}^4\right)^{\otimes N}$, $x_k \in \mathbb{R}^{1,3}$, $k = 1, \dots, N$
- In the non-interacting case Ψ satisfies a system of N Dirac eqs.:

$$\boxed{-i\hbar\gamma_k^\mu \partial_{x_k^\mu}\Psi + m\Psi = 0, \qquad k = 1, \dots, N} \tag{1}$$

- $\gamma_k^\mu = I \otimes \cdots \otimes \gamma^\mu \otimes \cdots \otimes I$, ($\gamma^\mu$ at the k-th place.)
- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \eta^{\mu\nu} I_{4\times 4}$, where $\eta = \mathrm{diag}(1, -1, -1, -1)$
- $N = 1$ case: 1-particle Dirac current $j^\mu = \bar{\psi}\gamma^\mu\psi$, $\bar{\psi} := \psi^\dagger\gamma^0$
- Note $j^0 = \psi^\dagger\psi > 0$ and $|\mathbf{j}| \leq j^0$ thus j^μ is future timelike.
- $-i\hbar\gamma^\mu\partial_\mu\psi + m\psi = 0$ implies $\partial_\mu j^\mu = 0$ i.e. $\partial_t j^0 + \partial_k j^k = 0$
- $\rho := j^0$, $v^k := j^k/j^0$ implies $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$ (continuity eq.)
- One-particle guiding law: $\boxed{\dfrac{d\mathbf{Q}}{dt} = \dfrac{\mathbf{j}(t, \mathbf{Q}(t))}{\rho(t, \mathbf{Q}(t))} = \mathbf{v}(t, \mathbf{Q}(t))}$.
- $N \geq 2$ case: N-particle current: $j^{\mu_1\cdots\mu_N} = \overline{\Psi}\gamma_1^{\mu_1}\dots\gamma_N^{\mu_N}\Psi$
- (1) implies $\partial_{x_k^{\mu_k}} j^{\mu_1\cdots\mu_k\cdots\mu_N} = 0$, $k = 1, \dots, N$.

Hypersurface Bohm-Dirac Theory (**DGMZ**, 1999)

- Let $\{\Sigma_\tau\}_{\tau\in\mathbb{R}}$ be a foliation of spacetime generated by unit timelike vectorfield $n^\mu(x)$. (Where would it come from? We shall see.)
- HBD guiding law:

$$\boxed{\frac{dX_k^\mu}{d\tau} = j^{\mu_1\ldots\overset{k\text{-th}}{\mu}\ldots\mu_N}(\mathbf{x}_1,\ldots,\mathbf{x}_N)\Pi_{j\neq k} n_{\mu_j}(\mathbf{x}_j)\Big|_{\mathbf{x}_\ell=\mathbf{X}_\ell(\tau)\in\Sigma_\tau,\ \ell=1,\ldots,N}}$$

- Example: on Minkowski space, with $\Sigma_t = \{x\in\mathbb{R}^{1,3} \mid x^0 = t\}$, we have $dX_k^\mu/d\tau = j^{0\ldots\overset{k\text{-th}}{\mu}\ldots0}(\mathbf{X}_1,\ldots,\mathbf{X}_N)$, where $\mathbf{X}_\ell = (t,\mathbf{Q}_\ell)$.
- Set $\rho := j^{0\ldots0}$. Then

$$\frac{dQ_k^i}{dt} = \frac{1}{\rho}\, j^{0\ldots\overset{k\text{-th}}{i}\ldots0}(t,\mathbf{Q}_1,\ldots,t,\mathbf{Q}_N) =: v_k^i(t;\mathbf{Q}_1,\ldots,\mathbf{Q}_N)$$

- Suppose the foliation is determined by the wave function of the system. Then the above can still be regarded as Lorentz covariant.
- So we can do relativistic quantum mechanics of N non-interacting spin-half particles.
- For interactions presumably we need photons, so how about spin-one particles?

Wave Function of a Single Photon

- Pauli alg.: $\mathbb{P} := \langle \{\sigma_k\}_{k=1}^3 \rangle = \mathrm{Cl}_{3,0}(\mathbb{R}) \cong M_2(\mathbb{C}) \cong \mathbb{C} \otimes \mathbb{H}$,
- Dirac alg.: $\mathbb{A} := \langle \{\gamma^\mu\}_{\mu=0}^3 \rangle_{\mathbb{C}} = \mathrm{Cl}_{1,3}(\mathbb{R})_{\mathbb{C}} \cong M_2(\mathbb{P}) \cong M_4(\mathbb{C})$
- Wave function of a spin-1/2 particle is a *rank-one bispinor*: $\psi \in \mathbb{C}^4$. (Note that $\mathbb{C}^4$ is a *minimal ideal* in $M_4(\mathbb{C})$.)
- Wave function of a spin-1 or spin-0 particle is a *rank-two bispinor*:

$$\boxed{x \to \Lambda x, \Lambda \in O(1,3) \implies \phi(x) \to \phi'(x) = \mathbf{L}_\Lambda \phi(\Lambda^{-1} x) \mathbf{L}_\Lambda^{-1}}$$

- ϕ is an $\mathbb{A}$-valued tensor field: $\phi = \begin{pmatrix} \phi_+ & \chi_- \\ \chi_+ & \phi_- \end{pmatrix}$, $\phi_\pm, \chi_\pm \in \mathbb{P}$.
- Trace and Trace-less parts: $\phi_\pm = (\frac{1}{2}\mathrm{tr}\,\phi_\pm)\mathbb{1} + \hat{\phi}_\pm$, so $\mathrm{tr}\,\hat{\phi}_\pm = 0$.
- Spin one: $\Phi = \begin{pmatrix} \phi_+ & \chi_- \\ \chi_+ & \phi_- \end{pmatrix}$, $\mathrm{tr}\,\phi_\pm = 0$. ($14\mathbb{C}$ off-shell.)
- Spin zero: $\phi = \begin{pmatrix} u_+ & \chi_- \\ \chi_+ & u_- \end{pmatrix}$, $u_\pm \in \mathbb{C}$. ($10\mathbb{C}$ off-shell.)
- "Spin degrees of freedom" for a spin-s particle is $2(2s+1)$.
- We will see that the *on-shell* degrees of freedom for these are 6 and 2, resp. (i.e. $\chi_\pm$ are determined by $\phi_\pm$,resp. $u_\pm$.)

Relativistic Wave Equation of the Electron and the Photon

- The Dirac operator $D := \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$, $D^\pm := \mathbb{1}\partial_t \pm \boldsymbol{\sigma} \cdot \nabla$.
- Dirac (1928): Wave equation for the electron (massive, spin-half) is

$$\boxed{-i\hbar D\psi_{el} + m_{el}\psi_{el} = 0,} \qquad \psi_{el} \in \mathbb{C}^4$$

- has conserved current $\nabla_\mu(\bar{\psi}_{el}\gamma^\mu\psi_{el}) = 0$
- M. Riesz (1946) generalized this to $\psi \in \mathbb{A}$ (for a massive, spin-one particle), found 4 new conservation laws $\nabla_\mu \mathrm{tr}\,(\bar{\psi}\gamma^\mu\psi\gamma^\nu) = 0$.
- Kiessling & T.-Z. (2018): Wave equation for the photon (massless spin-one particle with no longitudinal modes) is

$$\boxed{-i\hbar D\psi_{ph} + m_{\not{z}}\Pi\psi_{ph} = 0}$$

  where $\Pi : \mathbb{A} \to \mathbb{A}$ is projection onto diag. blocks, and $m_{\not{z}}$ is a constant needed for dimensional reasons (NOT a photon mass!)
- Thus $-i\hbar D^\pm\phi_\pm = 0$ & $-i\hbar D^\pm\chi_\mp + m_{\not{z}}\phi_\mp = 0$, so $\phi_\pm$ determine $\chi_\pm$ (except in one space dim.)

The Photon Probability Current and the Foliation

- Let $\phi_{\mathrm{PH}} := \Pi\psi_{\mathrm{PH}}$. Then $D\phi_{\mathrm{PH}} = 0$ (massless Dirac Eq.)
- Riesz tensor: $\tau^{\mu\nu} = \frac{1}{4}\mathrm{tr}\left(\overline{\phi_{\mathrm{PH}}}\gamma^\mu\phi_{\mathrm{PH}}\gamma^\nu\right)$, with $\overline{\phi_{\mathrm{PH}}} := \gamma^0\phi_{\mathrm{PH}}^\dagger\gamma^0$.
- $\boldsymbol{\tau}$ is symmetric, satisfies D.E.C., and $\nabla^\mu\tau_{\mu\nu} = 0$.
- $\pi^\mu := \int_{x^0=t}\tau^{\mu\nu}n_\nu d\mathbf{x}$.
- $\boldsymbol{\pi}$ is constant, future-directed and causal: $\pi^0 \geq \sqrt{\pi_i\pi^i}$.
- If $\boldsymbol{\pi}$ is not null (generically it won't be,) we set $\mathbf{X} := \frac{1}{\pi_\mu\pi^\mu}\boldsymbol{\pi}$.
- Photon probability current: $\boxed{j^\mu_{\mathrm{PH}} := \frac{1}{4}\mathrm{tr}\left(\overline{\phi_{\mathrm{PH}}}\gamma^\mu\phi_{\mathrm{PH}}\gamma_\nu X^\nu\right)}$.
- Σ_τ = foliation by hyperplanes dual to $\mathbf{n} := \mathbf{X}/\|\mathbf{X}\|$. It is determined *solely* by the initial wave function $\psi_{\mathrm{PH}}|_{x^0=0}$.
- Now we can use HBD to find photon trajectories!
- If $d = 1$, $\phi_{\mathrm{PH}} = \Pi\psi_{\mathrm{PH}} = 0$, but we can use ψ_{PH} itself to form $\tau^{\mu\nu}$, since $D\psi_{\mathrm{PH}} = 0$.

- The *one*-body wave functions for $d = 1$:
- $\psi_{\mathrm{PH}} = \begin{pmatrix} 0 & \chi_- \\ \chi_+ & 0 \end{pmatrix} \in V_2 \subset \mathbb{C}^{2\times 2}$ and $\psi_{\mathrm{EL}} = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} \in \mathbb{C}^2$
- The 2-body WF belongs to a 4-dim. tensor product space $V_2 \otimes \mathbb{C}^2$.
- $\psi = (\psi_{--}, \psi_{-+}, \psi_{+-}, \psi_{++}) : \mathcal{S} \to \mathbb{C}^4$, so $\psi = \psi(t_p, s_p, t_e, s_e)$
- Let $\gamma_p^\mu := \gamma^\mu \otimes \mathbb{1}$, $\gamma_e^\nu := \mathbb{1} \otimes \gamma^\nu$, $\mathbf{D}_p := \gamma_p^\mu \partial_{x_p^\mu}$, $\mathbf{D}_e := \gamma_e^\nu \partial_{x_e^\nu}$.
- The (free) *multi-time equations are:*

$$\begin{cases} -i\hbar \mathbf{D}_p \psi & = & 0 \\ -i\hbar \mathbf{D}_e \psi + m_e \psi & = & 0 \end{cases} \tag{2}$$

- *Note: this is ok because* $[\mathbf{D}_p, \mathbf{D}_e] = 0$.
- *We need to specify initial data:* $\psi(0, s_p, 0, s_e) = \mathring{\psi}(s_p, s_e)$.
- *We also need* boundary conditions *on the* coincidence set, *where* $t_p = t_e$ *and* $s_p = s_e$, *to prevent particles from "going through" each other.*

THEOREM (Lienert, Kiessling, & T.-Z., 2018)

- Let $\mathcal{S}_1 := \{(t_p, s_p, t_e, s_e) \in \mathcal{S} \mid s_p < s_e\}$. The following IBVP for the 2-body photon-electron multi-time wave function $\psi : \overline{\mathcal{S}_1} \to \mathbb{C}^4$,

$$\begin{cases} -i\hbar D_p \psi &= 0 & \\ -i\hbar D_e \psi + m_e \psi &= 0 & \text{in } \mathcal{S}_1 \\ \psi &= \mathring{\psi} \in C_c^\infty & \text{on } t_p = t_e = 0, s_p < s_e \\ \psi_{+-} &= e^{i\theta}\psi_{-+} & \text{on } t_p = t_e, s_p = s_e \end{cases}$$

  for any $\theta \in [0, 2\pi)$, has a unique global solution that is supported in $\overline{\mathcal{S}_1}$, depends continuously on the initial data $\mathring{\psi}$, is Lipschitz continuous everywhere.

- For typical (w.r.t. $|\mathring{\psi}|^2$) initial configurations of the particles, the Bohmian trajectories $Q_p(t)$ and $Q_e(t)$ for the photon and the electron exist globally, are unique, and $Q_p(t) \leq Q_e(t)$ for all $t > 0$.

Evolution of the 2-Particle Probability Density

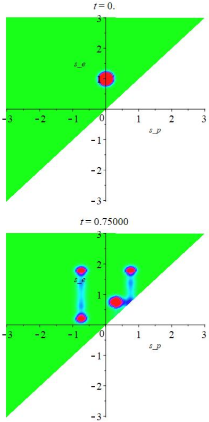
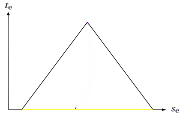
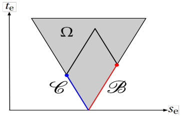


Figure: Density plot of ρ evolving in common time. The four peaks correspond to the four components of the joint wave function.

Method of Proof

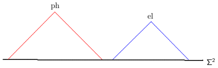
- The wavefunction in photonic variables satisfies a massless transport equation, so that
$$\Psi(t_{\text{ph}}, s_{\text{ph}}) = \mathcal{P}(t_{\text{ph}}, s_{\text{ph}})\mathring{\Psi} := \begin{pmatrix} 0 & \mathring{\chi}_-(s_{\text{ph}} - t_{\text{ph}}) \\ \mathring{\chi}_+(s_{\text{ph}} + t_{\text{ph}}) & 0 \end{pmatrix}.$$
- The wavefunction in electron variables satisfies KG equation, for which we can solve the Cauchy as well as the Goursat problems:
$$\begin{cases} \Box_{\text{e}}\Psi + \omega^2\Psi &= 0 \\ \Psi\big|_{t_{\text{e}}=0} &= A(s_{\text{e}}) \\ \partial_{t_{\text{e}}}\Psi\big|_{t_{\text{e}}=0} &= B(s_{\text{e}}) \end{cases}, \quad \begin{cases} \Box_{\text{e}}\Psi + \omega^2\Psi &= 0 \quad \text{for } |s_{\text{e}}| < t_{\text{e}} \\ \Psi(b, b) &= F(b) \\ \Psi(c, -c) &= G(c) \end{cases}$$
with Cauchy data (A, B) or characteristic data (F, G) that are derived from Dirac's eq. We thus have propagators
$$\Psi(t_{\text{e}}, s_{\text{e}}) = \mathcal{E}(t_{\text{e}}, s_{\text{e}})\mathring{\Psi}, \qquad \Psi(t_{\text{e}}, s_{\text{e}}) = G^R(t_{\text{e}}, s_{\text{e}})F + G^L(t_{\text{e}}, s_{\text{e}})G$$



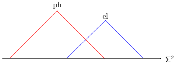


- Decompose $\mathcal{S}_1$ into two disjoint parts $\mathcal{S}_1 = \mathcal{F} \cup \mathcal{N}$ via

$$\mathcal{F} := \{(\mathbf{x}_{\mathrm{ph}}, \mathbf{x}_{\mathrm{e}}) \in \mathcal{S}_1 : s_{\mathrm{ph}} + t_{\mathrm{ph}} \leq s_{\mathrm{e}} - t_{\mathrm{e}}\}$$



$$\mathcal{N} := \{(\mathbf{x}_{\mathrm{ph}}, \mathbf{x}_{\mathrm{e}}) \in \mathcal{S}_1 : s_{\mathrm{ph}} + t_{\mathrm{ph}} > s_{\mathrm{e}} - t_{\mathrm{e}}\}$$



- $\mathcal{F}$ is governed only by the free equations, so that

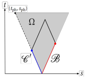
$$\Psi(\mathbf{x}_{\mathrm{ph}}, \mathbf{x}_{\mathrm{e}}) = \big(\mathcal{P}(\mathbf{x}_{\mathrm{ph}}) \otimes \mathcal{E}(\mathbf{x}_{\mathrm{e}})\big)\mathring{\Psi}.$$

- Fix $(\mathbf{x}_{\mathrm{ph}}, \mathbf{x}_{\mathrm{e}}) \in \mathcal{N}$. Above is still valid for ψ_{--} and ψ_{-+}:

$$\psi_{-\varsigma_1}(\mathbf{x}_{\mathrm{ph}}, \mathbf{x}_{\mathrm{e}}) = \Pi_{-\varsigma_1}\big(\mathcal{P}(\mathbf{x}_{\mathrm{ph}}) \otimes \mathcal{E}(\mathbf{x}_{\mathrm{e}})\big)\mathring{\Psi}.$$

- For ψ_{+-} solve a Goursat problem and use the boundary condition:

$$\psi_{+-}(\mathbf{x}_{\mathrm{ph}}, \mathbf{x}_{\mathrm{e}}) = G^R(t_{\mathrm{e}}, \tilde{s})\big(\psi_{+-}\big|_{\mathcal{B}}\big) + G^L(t_{\mathrm{e}}, \tilde{s})\big(e^{i\theta}\psi_{-+}\big|_{\mathcal{C}}\big)$$

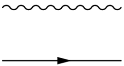


- For ψ_{++} solve a sourced transport equation:

$$\psi_{++}(\mathbf{x}_{\mathrm{ph}}, \mathbf{x}_{\mathrm{e}}) = \psi_{++}(\mathbf{x}_{\mathrm{ph}}, 0, s_{\mathrm{e}} + t_{\mathrm{e}}) - i\omega \int_0^{t_{\mathrm{e}}} \psi_{+-}(\mathbf{x}_{\mathrm{ph}}, \tau, s_{\mathrm{e}} + t_{\mathrm{e}} - \tau)\, d\tau$$

- All this can be nicely packaged using Feynman diagrams: In the "Far" region electron and photon are freely propagating:

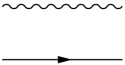
$$\Psi\Big|_{\mathcal{F}} \quad = $$



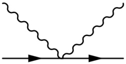
$$= \quad (\mathcal{P}\otimes\mathcal{E})\mathring{\Psi}$$

- In the near region there are two diagrams to sum over: freely propagated data, and data resulting from a collision:

$$\Psi\Big|_{\mathcal{N}} \quad = $$



$$+ \quad e^{i\theta}$$



$$= \begin{pmatrix} \Pi_{--}(\mathcal{P}\otimes\mathcal{E})\mathring{\Psi} \\ \Pi_{-+}(\mathcal{P}\otimes\mathcal{E})\mathring{\Psi} \\ G^R\left(\psi_{+-}\big|_{\mathcal{B}}\right) \\ \psi_{++}\big|_{\mathcal{B}} - i\omega\int_0^{t_e} G^R\left(\psi_{+-}\big|_{\mathcal{B}}\right) \end{pmatrix} + e^{i\theta}\begin{pmatrix} 0 \\ 0 \\ G^L\left(\psi_{-+}\big|_{\mathcal{C}}\right) \\ -i\omega\int_0^{t_e} G^L\left(\psi_{-+}\big|_{\mathcal{C}}\right) \end{pmatrix}$$

- Once we have the wavefunction everywhere, we can use the Hypersurface Bohm-Dirac theory to find the trajectories.

Interacting vs. Noninteracting Particle Trajectories

- Guiding equations (ODEs):

$$\frac{dQ_p(t)}{dt} = \frac{j^{10}(Q_p(t), Q_e(t))}{j^{00}(Q_p(t), Q_e(t))}, \qquad \frac{dQ_e(t)}{dt} = \frac{j^{01}(Q_p(t), Q_e(t))}{j^{00}(Q_p(t), Q_e(t))}$$

- where

$$\begin{aligned} j^{00} &= |\psi_{--}|^2 + |\psi_{-+}|^2 + |\psi_{+-}|^2 + |\psi_{++}|^2 \\ j^{01} &= |\psi_{--}|^2 - |\psi_{-+}|^2 + |\psi_{+-}|^2 - |\psi_{++}|^2 \\ j^{10} &= |\psi_{--}|^2 + |\psi_{-+}|^2 - |\psi_{+-}|^2 - |\psi_{++}|^2 \end{aligned}$$

- Existence and uniqueness of trajectories since ψ is locally Lipschitz.

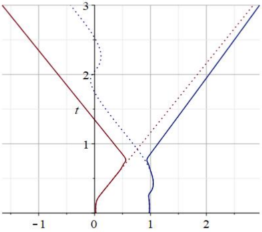
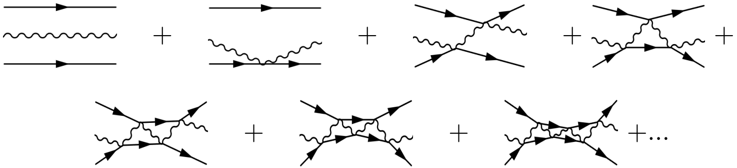


Figure: interacting (solid) versus non-interacting (dotted) trajectories

Two Electrons and One Photon (with L. Frolov & S. Leigh)

- $\Psi = \left(\psi_{\zeta_p\zeta_1\zeta_2}(t_p, s_p, t_1, s_1, t_2, s_2)\right)$, $\zeta_p, \zeta_1, \zeta_2 \in \{+, -\}$.
- Solves a system of two massive and one massless Dirac equations.
- Boundary conditions on the two coincidence sets: C_1 and C_2.
- New issue: photon can bounce infinitely often between 2 electrons!
- Possible divergence of the wave function at triple points!
- Remedy: *Leaky boundary conditions:* allow probability to leak out in an ϵ-nbhd of the triple point line. Find solution, then let $\epsilon \to 0$.
- Dissipative operators and contraction semi-groups, instead of self-adjoint operators and unitary evolution
- THEOREM (**Frolov, Leigh, & T.-Z.**, 2022) Global existence of 3-body multi-time wavefunction that preserves probabilities.
- Solution viewed as a convergent infinite sum of Feynman diagrams:



Some Details of the Proof

- Existence of a solution that is a continuous wave function with bounded derivatives is guaranteed by a result of Lienert & Nickel (2020) using a contraction mapping argument.
- We momentarily restrict to the equal time setting. (It is possible to recover the multi-time dynamics from this.)
- The single-time dynamics take place on $\mathcal{S}_1 := \{(s_{\mathrm{ph}}, s_{\mathrm{e}_1}, s_{\mathrm{e}_2}) : s_{\mathrm{e}_1} < s_{\mathrm{ph}} < s_{\mathrm{e}_2}\} \subset \mathbb{R}_{\mathrm{ph}} \times \mathbb{R}_{\mathrm{e}_1} \times \mathbb{R}_{\mathrm{e}_2}$
- Our goal is to show that for all $\mathring{\Psi} \in L^2(\mathcal{S}_1)$, the IBVP

$$\begin{cases} i\hbar\partial_t \Psi &= \hat{H}\Psi & \text{in } \mathbb{R}^+ \times \mathcal{S}_1 \\ \Psi\big|_{t=0} &= \mathring{\Psi} & \\ \psi_{-+\varsigma_2} &= e^{i\theta_1}\psi_{+-\varsigma_2} & \text{on } \{s_{\mathrm{ph}} = s_{\mathrm{e}_1}\} \\ \psi_{+\varsigma_1-} &= e^{i\theta_2}\psi_{-\varsigma_1+} & \text{on } \{s_{\mathrm{ph}} = s_{\mathrm{e}_2}\} \end{cases} \tag{3}$$

  has a unique, global-in-time solution expressible as an infinite diagram sum.

$$\hat{H} := \left(\hat{H}_{\mathrm{ph}} \otimes \mathbb{1} \otimes \mathbb{1}\right) + \left(\mathbb{1} \otimes \hat{H}_{\mathrm{e}_1} \otimes \mathbb{1}\right) + \left(\mathbb{1} \otimes \mathbb{1} \otimes \hat{H}_{\mathrm{e}_2}\right) \tag{4}$$

$$\hat{H}_{\mathrm{ph}} := -i\hbar\gamma^0\gamma^1\partial_{s_{\mathrm{ph}}}, \quad \hat{H}_{\mathrm{e}_j} := \gamma^0(m_{\mathrm{e}} - i\hbar\gamma^1\partial_{s_{\mathrm{e}_j}}). \tag{5}$$

Modify the IBVP!

Theorem (Leaky IBVP)

Let $\mu_\epsilon(x) : \mathbb{R} \to [0,1]$ *be smooth with* $\mu\big|_{[0,\epsilon)} = 0$, $\mu\big|_{[2\epsilon,\infty)} = 1$.

Modify the IBVP!

Theorem (Leaky IBVP)

Let $\mu_\epsilon(x) : \mathbb{R} \to [0,1]$ *be smooth with* $\mu\big|_{[0,\epsilon)} = 0$*,* $\mu\big|_{[2\epsilon,\infty)} = 1$*. Then for each* $\epsilon > 0$*, and* $\theta_1, \theta_2 \in [0, 2\pi)$*, the IBVP*

Modify the IBVP!

Theorem (Leaky IBVP)

Let $\mu_\epsilon(x) : \mathbb{R} \to [0,1]$ be smooth with $\mu\big|_{[0,\epsilon)} = 0$, $\mu\big|_{[2\epsilon,\infty)} = 1$. Then for each $\epsilon > 0$, and $\theta_1, \theta_2 \in [0, 2\pi)$, the IBVP

$$\begin{cases} i\hbar\partial_t \Psi = \hat{H}\Psi & \text{in } \mathbb{R}^+ \times \mathcal{S}_1 \\ \Psi\big|_{t=0} = \mathring{\Psi} & \\ \psi_{-+\varsigma_2} = e^{i\theta_1}\mu_\epsilon(|s_{e_1} - s_{e_2}|)\psi_{+-\varsigma_2} & \text{on } \{s_{ph} = s_{e_1}\} \\ \psi_{+\varsigma_1-} = e^{i\theta_2}\mu_\epsilon(|s_{e_1} - s_{e_2}|)\psi_{-\varsigma_1+} & \text{on } \{s_{ph} = s_{e_2}\} \end{cases} \tag{6}$$

admits a unique, global in time solution $\Psi_\epsilon(t)$ for all $\mathring{\Psi}$ bounded and compactly supported in $\mathcal{S}_1$.

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admits a unique, global in time solution $\Psi_\epsilon(t)$ for all $\mathring{\Psi}$ bounded and compactly supported in $\mathcal{S}_1$. Physically, this modification cuts out all diagrams where a bounce occurs while the electrons are too close. In region $\{s_{e_2} - s_{e_1} < \epsilon\}$ probability is "leaking through boundary".

Modify the IBVP!

Theorem (Leaky IBVP)

Let $\mu_\epsilon(x) : \mathbb{R} \to [0,1]$ be smooth with $\mu\big|_{[0,\epsilon)} = 0$, $\mu\big|_{[2\epsilon,\infty)} = 1$. Then for each $\epsilon > 0$, and $\theta_1, \theta_2 \in [0, 2\pi)$, the IBVP

$$\begin{cases} i\hbar\partial_t \Psi &= \hat{H}\Psi & \text{in } \mathbb{R}^+ \times \mathcal{S}_1 \\ \Psi\big|_{t=0} &= \mathring{\Psi} & \\ \psi_{-+\varsigma_2} &= e^{i\theta_1}\mu_\epsilon(|s_{e_1} - s_{e_2}|)\psi_{+-\varsigma_2} & \text{on } \{s_{ph} = s_{e_1}\} \\ \psi_{+\varsigma_1-} &= e^{i\theta_2}\mu_\epsilon(|s_{e_1} - s_{e_2}|)\psi_{-\varsigma_1+} & \text{on } \{s_{ph} = s_{e_2}\} \end{cases} \tag{6}$$

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Generator of evolution

Theorem ((Lumer-) Phillips (1959))

Let $U(t)$ be a C_0 contraction semigroup on $L^2(\mathcal{S}_1)$. Then there exists a linear operator B such that

- $D(B) := \{\Psi \in L^2(\mathcal{S}_1) : \frac{d}{dt} U(t)\Psi\big|_{t=0}$ *exists in* $L^2(\mathcal{S}_1)\}$ *is dense in* $L^2(\mathcal{S}_1)$
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We denote the generator of $U_\epsilon(t)$ as $\frac{-i}{\hbar}\hat{H}_\epsilon$. Note $\hat{H}_\epsilon$ extends the *minimal* operator $\hat{H}_{\min} := \hat{H}\big|_{C_c^\infty(\mathcal{S}_1)}$ since

$$\frac{d}{dt} U_\epsilon(t)\mathring{\Psi}\Big|_{t=0} = \frac{-i}{\hbar}\hat{H}\mathring{\Psi}, \quad \forall \mathring{\Psi} \in C_c^\infty(\mathcal{S}_1). \tag{7}$$

Maximally dissipative extensions

Key idea: Maximally dissipative extensions of $\frac{-i}{\hbar}\hat{H}_{\text{min}}$ correspond to the placement of boundary conditions along $\partial\mathcal{S}_1$.

Theorem (Arendt, Chalendar, and Eymard (2023))

All dissipative extensions of $\frac{-i}{\hbar}\hat{H}_{min}$ are restrictions of the maximal operator $\frac{-i}{\hbar}\hat{H}^*_{min}$. In particular, there is a 1 to 1 correspondence between maximally dissipative extensions B of $\frac{-i}{\hbar}\hat{H}_{min}$ and linear contractions $T_B : \ker(i + \hat{H}^*_{min}) \to \ker(i - \hat{H}^*_{min})$ in that $D(B)$ must be of form

$$D(B) = \{\Psi \in D(\hat{H}^*_{min}) : \Psi_+ = T_B \Psi_-\} \tag{8}$$

Here $\Psi = \Psi_{min} + \Psi_+ + \Psi_- \in D(\hat{H}^*_m)$ has been decomposed via

$$D(\hat{H}^*_{min}) = D(\overline{H_{min}}) \oplus \ker(i - \hat{H}^*_{min}) \oplus \ker(i + \hat{H}^*_{min}) \tag{9}$$

Claim: Setting $\Psi_+ = T_B\Psi_-$ is equivalent to setting a boundary condition!

Boundary Conditions

- Consider $\hat{H}_{\min} = i\partial_x$ defined on on $C_c^\infty([-1,1],\mathbb{C}) \subset L^2([0,1],\mathbb{C})$.

$$\ker(i \mp \hat{H}^*_{\min}) = \{C_0 e^{1\pm x} : C_0 \in \mathbb{C}\} \tag{10}$$

- When we decompose $\Psi \in D(\hat{H}^*_{\min})$, the functions $\Psi_\pm$ are uniquely determined by the values of Ψ at $x = -1$ and $x = 1$

$$\Psi_+(x) = \Psi(-1)e^{1+x}, \quad \Psi_-(x) = \Psi(1)e^{1-x} \tag{11}$$

- Setting $\Psi_+ = T_B\Psi_-$ sets a relationship between $\Psi(-1)$ and $\Psi(1)$!
- A more complicated version of this argument applied to our problem allows us to explicitly construct the linear contractions T_ϵ corresponding to the boundary conditions

$$\begin{cases} \psi_{-+\varsigma_2} = e^{i\theta_1}\mu_\epsilon(|s_{e_1} - s_{e_2}|)\psi_{+-\varsigma_2} & \text{on } \{s_{ph} = s_{e_1}\} \\ \psi_{+\varsigma_1-} = e^{i\theta_2}\mu_\epsilon(|s_{e_1} - s_{e_2}|)\psi_{-\varsigma_1+} & \text{on } \{s_{ph} = s_{e_2}\} \end{cases} \tag{12}$$

Convergence of evolutions

After verifying that $T_\epsilon \to T_0$ as $\epsilon \to 0$, we can apply an approximation theorem of Kurtz to prove convergence of the evolutions

Theorem (Kurtz (1969))

Let $T_\epsilon : \ker(i + \hat{H}^*_{min}) \to \ker(i - \hat{H}^*_{min})$ be a family of linear contractions and $\frac{-i}{\hbar}\hat{H}_\epsilon$ their associated maximally dissipative extensions of $\frac{-i}{\hbar}\hat{H}_{min}$. If T_ϵ converges to T_0 as $\epsilon \to 0$, then the generated evolutions $\exp(\frac{-i}{\hbar}t\hat{H}_\epsilon)$ converge to $\exp(\frac{-i}{\hbar}t\hat{H}_0)$ as $\epsilon \to 0$.

Corollary

For $\epsilon > 0$ and $\mathring{\Psi} \in L^2(\mathcal{S}_1)$, let $\Psi_\epsilon(t)$ denote the solution to the "Leaky IBVP" with initial data $\mathring{\Psi}$. Then $\Psi_\epsilon(t)$ converges as $\epsilon \to 0$ to $\Psi_0(t)$ solving the IBVP with boundary condition

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- FUTURE TASK: Investigate the role of θ_1, θ_2 in the dynamics!

Summary

- Hypersurface Bohm-Dirac (HBD) theory is a relativistic extension of Bohmian Mechanics in which given a foliation of spacetime, one can use the conserved tensor current associated with the multi-time wave function of a system of N particles to find the worldlines of those particles.
- Photons too, can be viewed as particles, guided by a Clifford algebra-valued wavefunction that satisfies a Dirac-type equation.
- One can use the conserved Riesz tensor of this wavefunction to single out a particular foliation of the Minkowski spacetime by hyperplanes, and use the HBD theory to find photon trajectories.
- In one space dimension, interaction of photons and electrons can in principle be entirely understood in terms of contact interactions, which are modeled by imposing a boundary condition on the wave function that ensures probability conservation.

Current and Future Directions

- The emergence of electric charge and the role of the phase shift parameters in the boundary conditions
- The possibility of extension to three space dimensions; evading Svendsen!
- Emission and Absorption phenomena
- Particles as singularities of fields with a quantum law of motion
- Ring-like particles and branched spacetimes
- Quantum dynamics on classical curved background spacetimes
- Covariant guiding laws for electromagnetic and gravitational fields

- **THERE HAS NEVER BEEN A MORE EXCITING TIME TO WORK ON FUNDAMENTAL PHYSICS!**

THANK YOU FOR LISTENING!