Triviality of mean-field φ^4 theories in four dimensions

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Outline

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  - ➔ Landau pole : divergence of the running coupling constant $g(\lambda)$ at a certain energy.
  - ➔ If $\lambda_{\max} \longrightarrow +\infty$ and $g(\lambda_{\max})$ fixed: finite result only if $g_{ren} = g(\lambda_{phys}) \underset{\lambda_{\max}\to+\infty}{\longrightarrow} 0$: **Theory = trivial** .

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- Triviality of the φ_d^4 theory in d dimensions:
  - ➔ $d > 4$: Triviality of the continuum limit on a lattice, by Aizenman and Fröhlich in [1, 2] only if $N = 1, 2$ [1982].
  - ➔ $d = 4$: Multi-scale analysis by Aizenman and Duminil-Copin in [3] [2021].

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- Discussions about the consequences of the triviality : upper bounds on the Higgs mass before its discovery in 2012.
- Our method is based on the flow equations [4] : analysis of the $O(N)$-model, $N \geq 1$.

Triviality of the mean-field $O(N)$-model scalar theory in four dimensions

The $O(N)$-model

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- Scalar field with N real components:

$$\varphi(x) = \left[\varphi_1(x), \cdots, \varphi_N(x)\right]^T \ .$$

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- Scalar field with N real components:

$$\varphi(x) = \left[\varphi_1(x), \cdots, \varphi_N(x)\right]^T \ .$$

- Consider a theory with a global $O(N)$-symmetry, invariant under $\varphi \mapsto -\varphi$: only even moments are non-vanishing.

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$$\int_p := \int \frac{d^4p}{(2\pi)^4}, \quad \hat{f}(p) = \int_p e^{ipx} f(x) \ . \tag{1}$$

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- Regularized propagator in momentum space:

$$C_{ij}^{\alpha_0,\alpha}(p,m) = \delta_{ij} \frac{1}{p^2+m^2} \Big(\exp(-\alpha_0(p^2+m^2)) - \exp(-\alpha(p^2+m^2)) \Big) \ , \tag{2}$$

with m the mass, α_0: UV-cutoff, $\alpha \in [\alpha_0, +\infty)$: flow parameter

Bare interaction lagrangian

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- Bare interaction lagrangian for a theory with a global $O(N)$-symmetry:

$$L^N_{0,\mathcal{V}}(\varphi) = \int_{\mathcal{V}} d^4x \Big[\sum_{n\in 2\mathbb{N}} c_{0,n}(\alpha_0)\varphi^n(x) \Big] \,. \qquad (3)$$

with $\varphi^2(x) := \sum_{1\leq i\leq N} \varphi_i^2(x)$ and $\varphi^n(x) = (\varphi^2(x))^{\frac{n}{2}}$ for $n \in 2\mathbb{N}$.

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- $c_{0,n}(\alpha_0)$: bare couplings $\rightarrow$ relevant/marginal for $n = 2, 4$, irrelevant for $n \geq 6$. No wavefunction renormalization term because of the mean-field limit.
- $\mathcal{V}$: finite volume in $\mathbb{R}^4$.

Correlation functions

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✳ Correlation functions in the finite volume $\mathcal{V}$

$$\langle \varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n)\rangle_{N,\mathcal{V}}^{\alpha_0,\alpha} := \frac{1}{Z_{N,\mathcal{V}}^{\alpha_0,\alpha}} \int d\mu_{N,\mathcal{V}}^{\alpha_0,\alpha}(\varphi) e^{-L_{0,\mathcal{V}}^{N}(\varphi)} \varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n) \, , \tag{4}$$

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- Generating functional of the Connected Amputated Schwinger (CAS) functions $L_{N,\mathcal{V}}^{\alpha_0,\alpha}(\varphi)$

$$-L_{N,\mathcal{V}}^{\alpha_0,\alpha}(\varphi) := \log\left(\int d\mu_{N,\mathcal{V}}^{\alpha_0,\alpha}(\psi)\exp(-L_{0,\mathcal{V}}^{N}(\varphi+\psi)\right) - \log(Z_{N,\mathcal{V}}^{\alpha_0,\alpha})\ . \tag{5}$$

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- Expansion in a power series of $\hat{\varphi}_i$ of the generating functional of the CAS function

$$L_N^{\alpha_0,\alpha}(\varphi) = \sum_{n\in 2\mathbb{N}} \sum_{1\leq i_1,\cdots,i_n\leq N} \int_{p_1,p_2,\cdots,p_n} \bar{\mathcal{L}}^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n}(p_1,\cdots,p_n)\hat{\varphi}_{i_1}(p_1)\cdots\hat{\varphi}_{i_n}(p_n). \tag{6}$$

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- Factorization due to translation invariance in position space

$$\bar{\mathcal{L}}^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n}(p_1,\cdots,p_n) = \underbrace{\delta^4\left(\sum_{i=1}^{n} p_i\right)}_{\text{distributional}} \underbrace{\mathcal{L}^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n}(p_1,\cdots,p_n)}_{\text{smooth}} . \quad (7)$$

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- Further **technical** simplification : *set* $m = 0$ *and restrict the study to* $\alpha \in [\alpha_0, 1]$ (units such that $m^2 = 1$, artificial IR cutoff).
- Define $A^{\alpha_0,\alpha}_{n;i_1 i_2 \cdots i_n} := \mathcal{L}^{\alpha_0,\alpha}_{n;i_1 i_2 \cdots i_n}(0, \cdots, 0)$.

Flow equations in the m.f.a.

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$$\partial_\alpha A^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n} = \binom{n+2}{2}\frac{c}{\alpha^2}\sum_{j=1}^{N} A^{\alpha_0,\alpha}_{n+2,i_1 i_2\cdots i_n jj} - \frac{1}{2}\sum_{n_1+n_2=n+2} n_1 n_2 \sum_{j=1}^{N}\mathbb{S}\left[A^{\alpha_0,\alpha}_{n_1;i_1 i_2\cdots i_{n_1-1}j} A^{\alpha_0,\alpha}_{n_2;i_{n_1} i_{n_1+1}\cdots i_n j}\right]; \tag{8}$$

with $c := \frac{1}{16\pi^2}$, $\mathbb{S}$ is an average operator permuting colour indices.

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  with $c := \frac{1}{16\pi^2}$, $\mathbb{S}$ is an average operator permuting colour indices.
- First difficulty: **no inductive scheme** so far : use the symmetry of the theory to analyze the dependence in the vector indices.

Properties of the mean-field CAS functions

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(P1): $A^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n} = 0$ if n is odd ($\mathbb{Z}_2$-symmetry).

(P2): $A^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n}$ is symmetric under any permutation of its indices $i_1, i_2, \cdots, i_n$ (Bose symmetry).

(P3): $A^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n}$ must be $O(N)$-invariant in the following sense:

Let O be an orthogonal matrix i.e. $O^T O = O^T O = I$, then

$$O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_n j_n} A^{\alpha_0,\alpha}_{n;j_1 j_2\cdots j_n} = A^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n} \,. \tag{9}$$

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- Properties **(P1)-(P2)** and **(P3)** imply

$$A^{\alpha_0,\alpha}_{n;i_1 i_2\cdots i_n} = A^{\alpha_0,\alpha}_n F_{i_1 i_2\cdots i_n} \ , \tag{10}$$

with $A^{\alpha_0,\alpha}_n$ smooth and

$$F_{i_1 i_2\cdots i_n} := \delta_{(i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{n-1} i_n)} = \frac{1}{n!}\sum_{\sigma\in S_n}\delta_{i_{\sigma(1)} i_{\sigma(2)}}\cdots\delta_{i_{\sigma(n-1)} i_{\sigma(n)}} \ . \tag{11}$$

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- Then we obtain

$$\partial_\alpha A^{\alpha_0,\alpha}_n = \binom{n+2}{2}\frac{N+n}{n+1}\frac{c}{\alpha^2}A^{\alpha_0,\alpha}_{n+2} - \frac{1}{2}\sum_{n_1+n_2=n+2} n_1n_2A^{\alpha_0,\alpha}_{n_1}A^{\alpha_0,\alpha}_{n_2} \ . \tag{12}$$

Remark: $N=1$ gives the single component case treated by Kopper in [4].

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- Factor out the power counting and the combinatorial factor

$$f_n(\mu) := n\,\alpha^{2-\frac{n}{2}} c^{\frac{n}{2}-1} A_n^{\alpha_0,\alpha}, \quad \mu := \ln\left(\frac{\alpha}{\alpha_0}\right) . \tag{13}$$

The variable μ lives in the compact $[0, \mu_{\max}]$ with $\mu_{\max} := \ln\left(\frac{1}{\alpha_0}\right)$.

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Flow equations in the m.f.a. (inductive form)

$$f_{n+2}(\mu) = \frac{2}{n(n+N)}\partial_\mu f_n(\mu) + \frac{n-4}{n(n+N)} f_n(\mu) + \frac{1}{n+N}\sum_{n_1+n_2=n+2} f_{n_1}(\mu) f_{n_2}(\mu) . \tag{14}$$

Construction of a trivial solution

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- Start with this bare interaction lagrangian

$$L_0^N(\varphi) = \int d^4x \left(c_{0,2}\varphi^2(x) + \underbrace{c_{0,4}}_{\geq 0} \varphi^4(x) \right) . \quad (15)$$

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- Boundary conditions in the m.f.a. at the bare level ($\mu = 0$):

$$f_2(0) = 2(2\pi)^4 \alpha_0 c_{0,2}, \quad f_4(0) = 4\pi^2 c_{0,4}, \quad f_n(0) = 0, \quad n \geq 6 \,. \tag{16}$$

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- IDEA: Construct the two-point function $f_2(\mu)$ such that
  - The solutions $f_n(\mu)$ constructed inductively from (14) are smooth and satisfy the boundary conditions (16).
  - The quantities $f_n(\mu_{\max})$ vanish when $\mu_{\max} \to +\infty$ i.e. $\alpha_0 \to 0$: Triviality of the mean-field theory !

Construction of a trivial solution

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- **Crucial ingredient**: ansatz of $f_2(\mu)$ introduced by Kopper [4] of the form

$$\sum_{n\geq 1} b_n \frac{x_n^{n-1}}{1+x_n^n}, \quad x_n := n\mu \,, \tag{17}$$

with $(b_n)_{n\geq 1}$ a sequence of real numbers so that the boundary conditions (16) are satisfied.

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with $(b_n)_{n\geq 1}$ a sequence of real numbers so that the boundary conditions (16) are satisfied.
- **Aim** : Prove that $f_2(\mu)$ is well defined for $\mu \in [0, \mu_{\max}]$.

Proposition. Geometric bounds for the coefficients b_n

There exists a constant $C(N, c_{0,2}, c_{0,4}) > 0$ such that
$$|b_n| \leq C(N, c_{0,2}, c_{0,4}) \frac{n^2}{2^n} \, , \quad n \geq 1 \, . \tag{18}$$

Proof is done by induction using (14) and is delicate (see [9]).

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- From (18) the solutions $f_n(\mu)$ are well-defined for $\mu \in [0, \mu_{\max}]$.
- The solutions $f_n(\mu)$ vanish in the UV-limit in the following sense:

$$\lim_{\mu_{\max} \to +\infty} \partial_\mu^l f_n(\mu_{\max}) = 0, \quad l \geq 0, \; n \geq 2 \,. \tag{19}$$

Proof is done by induction using (14) and the ansatz.

Theorem. Existence of a trivial mean-field solution

Consider a bare interaction lagrangian (15) and the corresponding mean-field boundary conditions (16) with $c_{0,2}$ and $c_{0,4}$ arbitrary constants. Then there exist smooth solutions of (14) which satisfy the boundary conditions (16) and vanish in the UV-limit.

Uniqueness of the trivial solution

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- **Non-trivial point**: the solutions $f_n(\mu)$ are not analytic at $\mu = 0$!
  Linked to the infinite set of constraints $f_n(0) = 0, \quad n \geq 6$.

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- For simplicity, take $N = 1$. Consider the mean field effective action

$$L_{mf}^{0,\mu}(x) = \sum_{n \in 2\mathbb{N}} A_n^{0,\mu} x^n \ . \tag{20}$$

Proposition. Local analyticity of the mean-field effective action

Consider the solutions $f_n(\mu)$ constructed from the ansatz (17) for the fixed boundary conditions (16). Then $L_{mf}^{0,0}(x)$ is polynomial w.r.t. x and for $\mu > 0$, the function $L_{mf}^{0,\mu}(x)$ is locally analytic w.r.t. x.

Idea of the proof: find bounds on the derivatives of $f_2(\mu)$, then use the FEs (14) to deduce inductive bounds on $f_n(\mu)$.

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- **Idea**: find an integral formulation of the mean-field FEs (14).

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- Inspired by Felder's idea : continuum limit of the hierarchical model [10] → Effective action u at scale $L^{-1}\lambda$, with $L > 1$, related to the effective action at scale λ by

$$e^{-u(L^{-1}\lambda,x)} = \int d\mu_L(y) e^{-L^4 u(\lambda, L^{-1}x+y)} , \quad \lambda \in (0, \Lambda_0] , \qquad (21)$$

where μ_L is the one-dimensional Gaussian measure defined by

$$d\mu_L(y) := \frac{1}{\sqrt{2\pi(L-1)}} e^{-\frac{y^2}{2(L-1)}} dy . \qquad (22)$$

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- Take the L-derivative of (21) and evaluate at $L = 1$:

$$-\lambda\partial_\lambda u = \frac{1}{2}\partial_{xx}u - \frac{1}{2}\left(\partial_x u\right)^2 + 4u - x\partial_x u \ . \tag{23}$$

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- Expansion of $u(\lambda, x)$ as a power series in x

$$u(\lambda, x) = \sum_{n\in 2\mathbb{N}} \frac{(2)^{\frac{n}{2}} f_n(\lambda)}{n} x^n \ , \tag{24}$$

and setting $\lambda = \Lambda_0 e^{-\frac{\mu}{2}}$, we obtain (14): Integral formulation of the mean-field FEs $\rightarrow$ Uniqueness of the solution u and its moments $f(\lambda)$ if u has a non-zero radius of convergence around x.

Uniqueness of the trivial solution

Theorem. Uniqueness of the trivial solution

For fixed mean-field boundary conditions (16), let $f_n(\mu)$ be the mean-field solutions of (14) constructed from the ansatz (17). Let $\tilde{f}_n(\mu)$ be solutions of the mean-field FE (14) which satisfy $f_n(0) = \tilde{f}_n(0)$. We assume that the corresponding mean-field effective action $\tilde{u}(\lambda, x)$ is locally analytic w.r.t. x for $\lambda < \Lambda_0$. Then $f_n(\mu) = \tilde{f}_n(\mu)$.

Extension to $N > 1$ can be done analogously (see [9] for more details).

Perspectives

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- Beyond the mean-field approximation ? Difficult to tackle this problem due to the momenta dependence.
- Can we study fermionic fields, gauge fields ?

Thank you for your attention

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